Microsoft

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Fault tolerant computation with four dimensional geometric codes

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Microsoft Quantum

References:
arXiv:2505.10403
arXiv:2506.15130

New opportunities for QEC code design

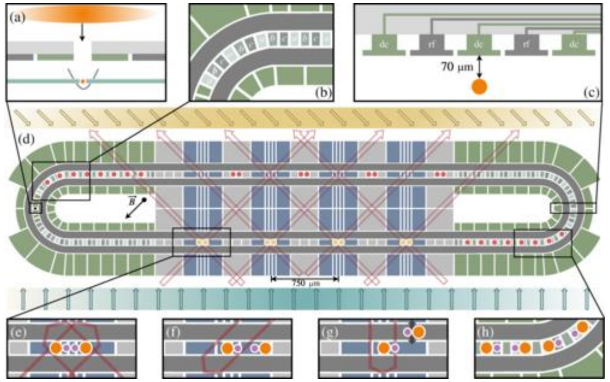
Rapid advances over the last decade in trapped ion and neutral atoms

Enabling all-to-all qubit connectivity

In parallel, we've seen major advances in high-performance quantum LDPC codes

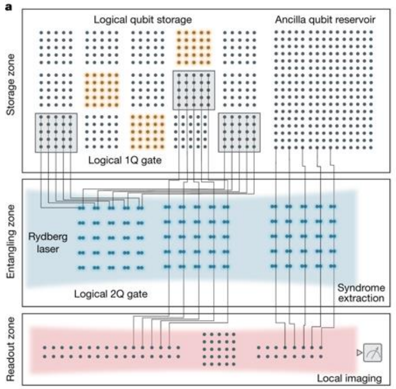
Platforms with all-to-all connectivity open the door to codes that were previously impractical due to geometric constraints

Quantinuum



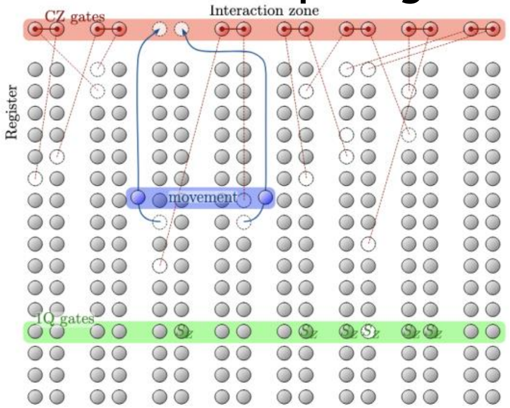
Phys. Rev. X 13, 041052 (2023)

Lukin Lab



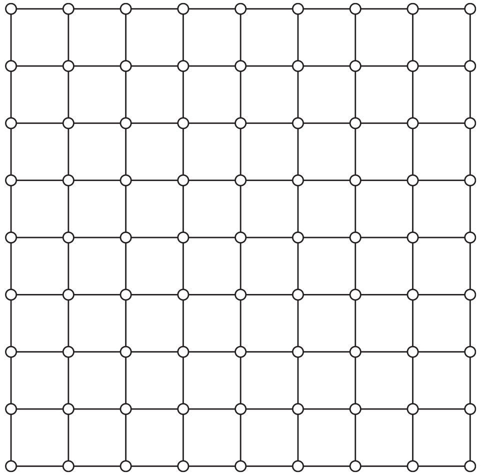
Nature 626, 58–65 (2024)

Atom computing

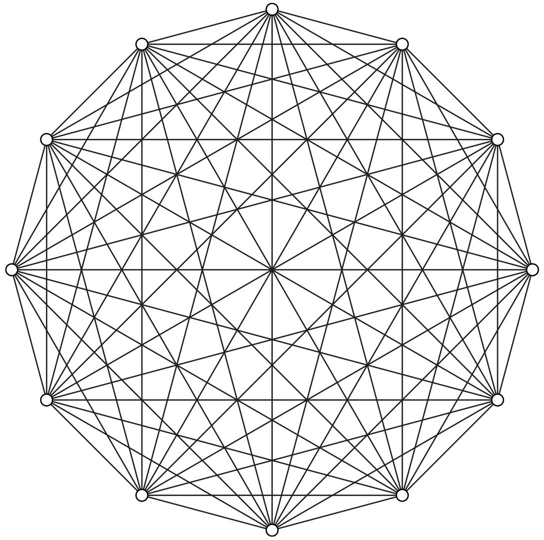


arXiv:2411.11822

Surface code



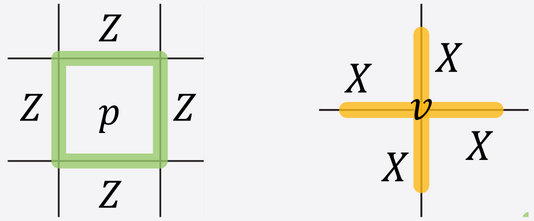
LDPC



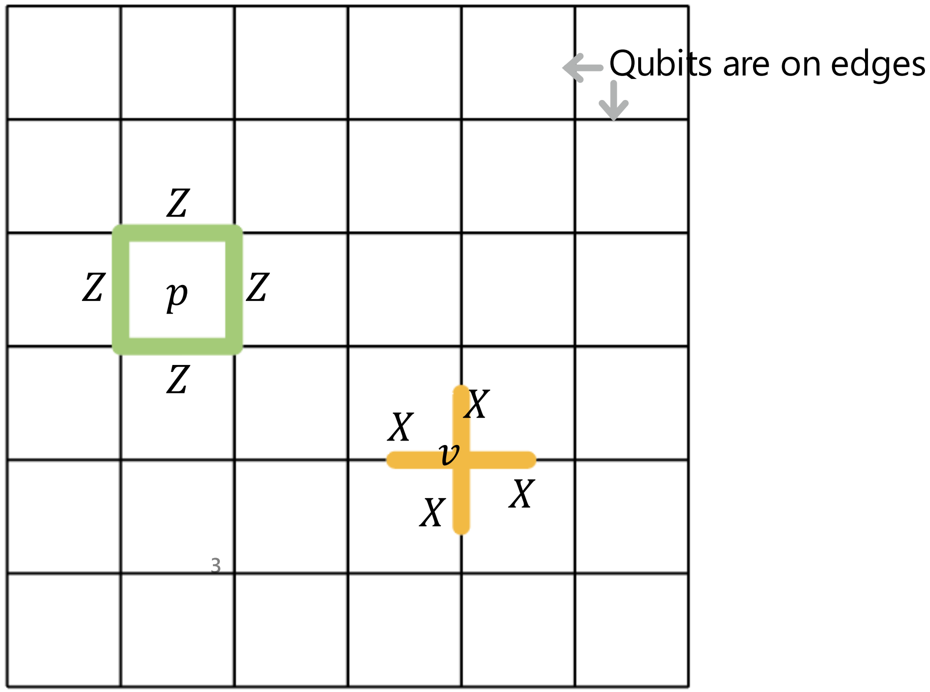
Review: Breuckmann, Eberhardt, PRX Quantum 2, 040101

Review of 2D Toric code

Stabilizers are on plaquettes and vertices



Code space is the +1 eigenspace of stabilizers



Space is a torus

A Yu Kitaev. Quantum error correction with imperfect gates. In Quantum communication, computing, and measurement, pages 181–188. Springer, 1997.

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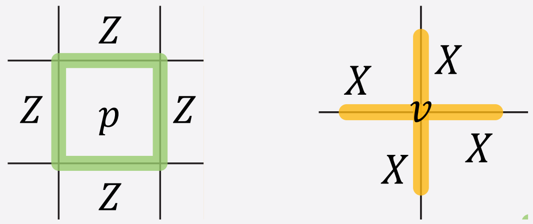
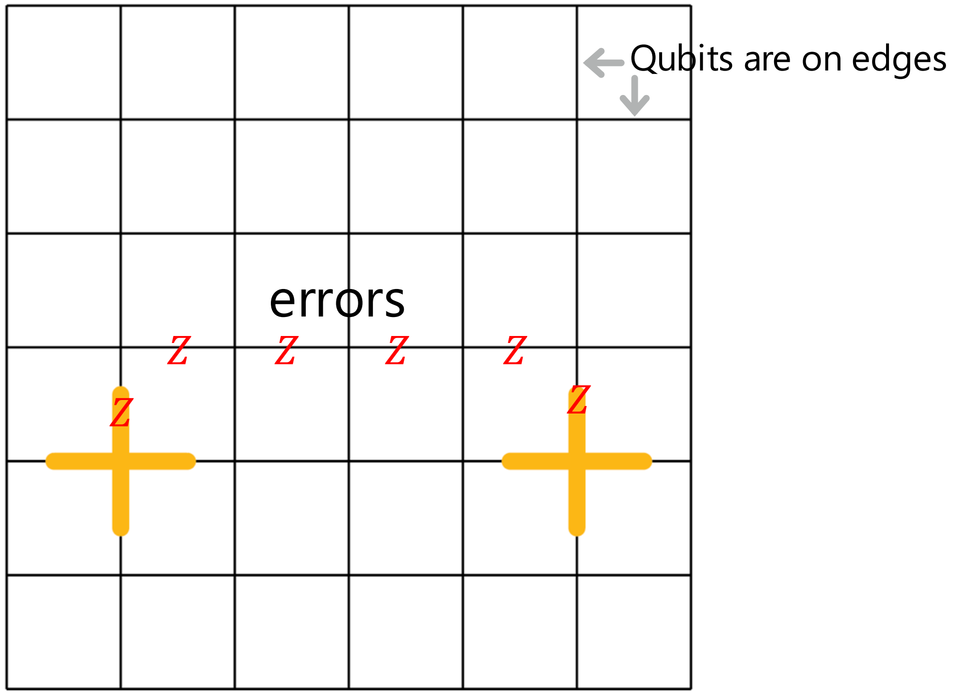


Code space is the +1 eigenspace of stabilizers

Z errors violate vertex stabilizers



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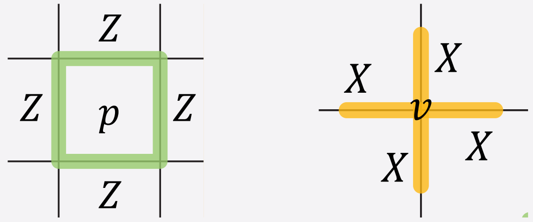
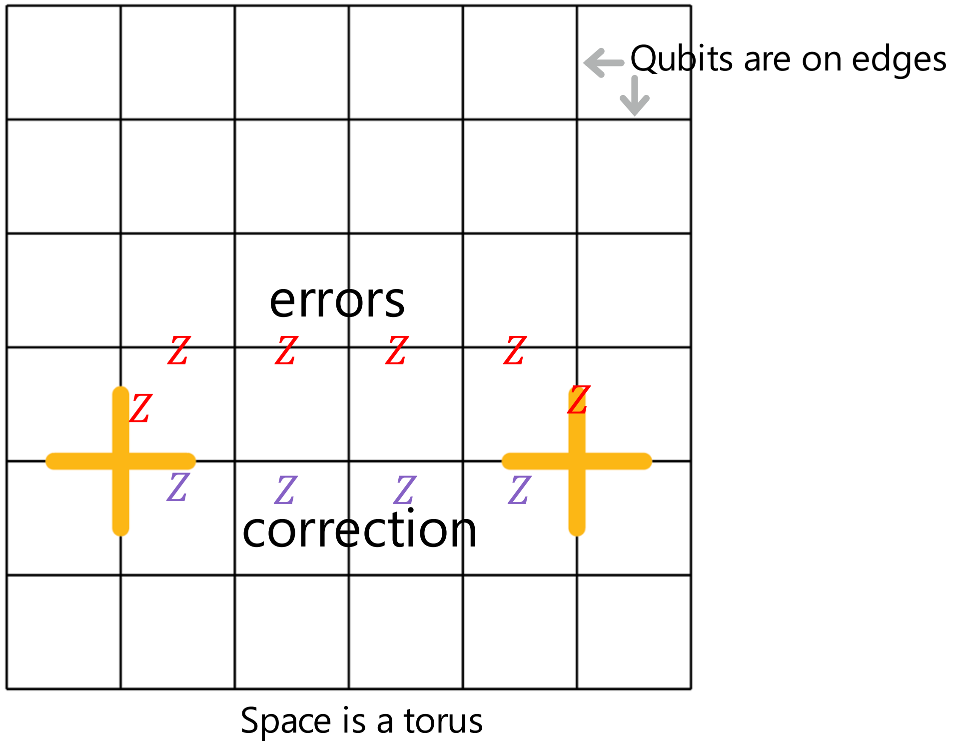


Code space is the +1 eigenspace of stabilizers

Z errors violate vertex stabilizers

Correct Z errors by minimum length path connecting violated X stabilizers



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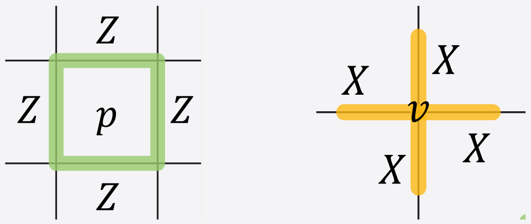


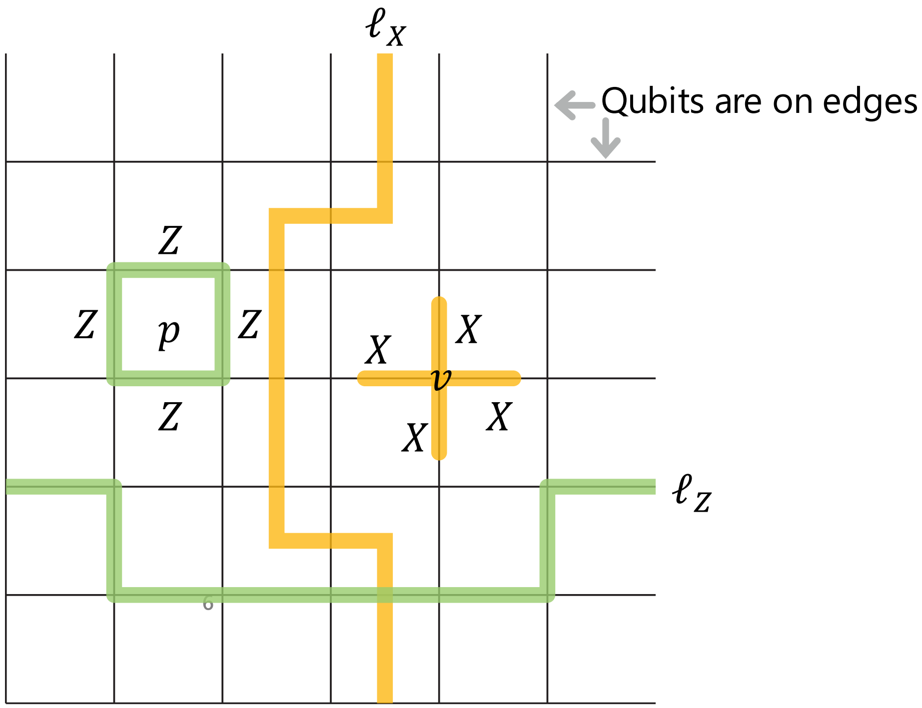
Code space is the +1 eigenspace of stabilizers

Logical operators are string operators:

- X-logicals are paths on dual lattice
- Z-logicals are paths on direct lattice

Code parameters $[[2d^2, 2, d]]$

d rounds of error correction for fault tolerance



Space is a torus

A Yu Kitaev. Quantum error correction with imperfect gates. In Quantum communication, computing, and measurement, pages 181–188. Springer, 1997.

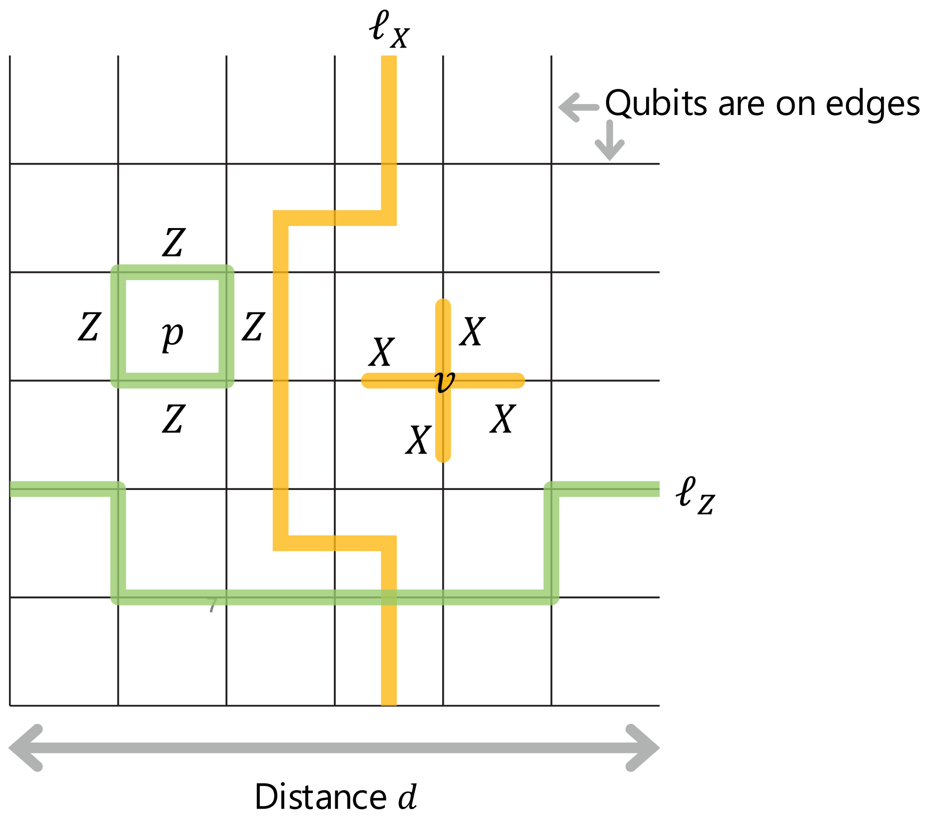
New opportunities for HW/QEC code design

Inputs:

- Hardware (e.g., trapped ions, neutral atoms, defect centers,...)
- Primitive operations (e.g., CZ, CNOT, S, H, MZ, etc)
- Hardware aware noise model (e.g., bias)
- Target application(s)
- Total number of qubits N=100,1000, 10000,...

Design [[n,k,d]] QEC code to:

- ☐ Maximize encoding rate → high-rate code, maximize k/n
- ☐ Minimize logical failure rate → high code distance
- ☐ Efficient logical cycle → single-shot error correction
- ☐ Maximize computational spacetime volume → efficient compilation of logical operations



A Yu Kitaev. Quantum error correction with imperfect gates. In Quantum communication, computing, and measurement, pages 181–188. Springer, 1997.

Review of 2D Toric code

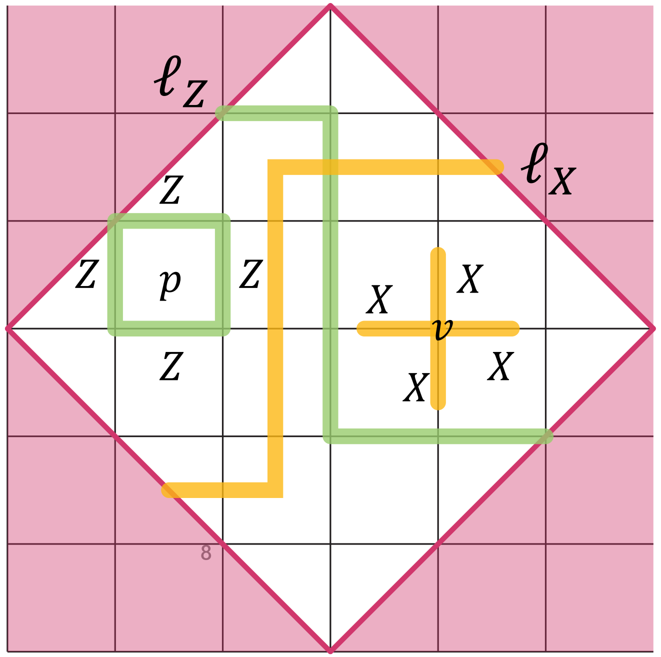
"Rotation" can cut qubit count in half while preserving code distance

Wen Phys. Rev. Lett. 90, 016803

Code parameters $[[2d^2, 2, d]] \rightarrow [[d^2, 2, d]]$

Key idea in this talk:

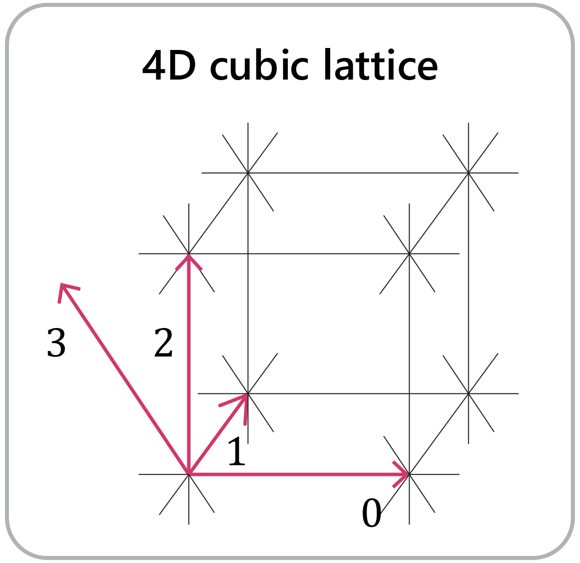
- extend rotation to 4D loop only toric code
- optimize over the space of rotated codes

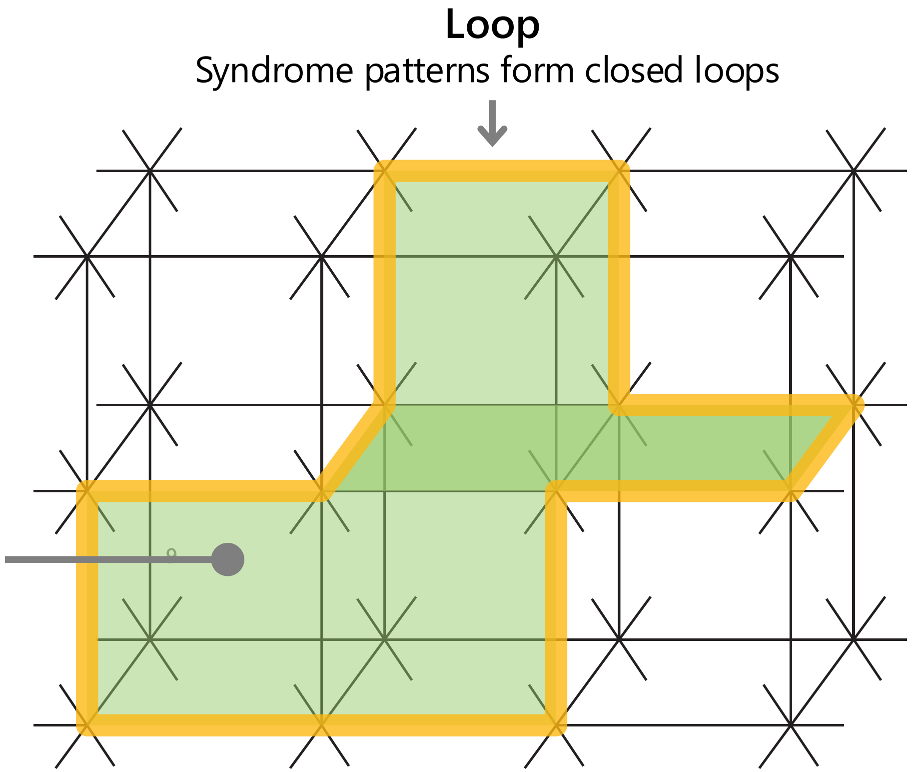


4D loop only Toric code on cubic lattice

Why 4D?

- First dimension for single-shot
- Lowest weight stabilizers in Toric family with single-shot property



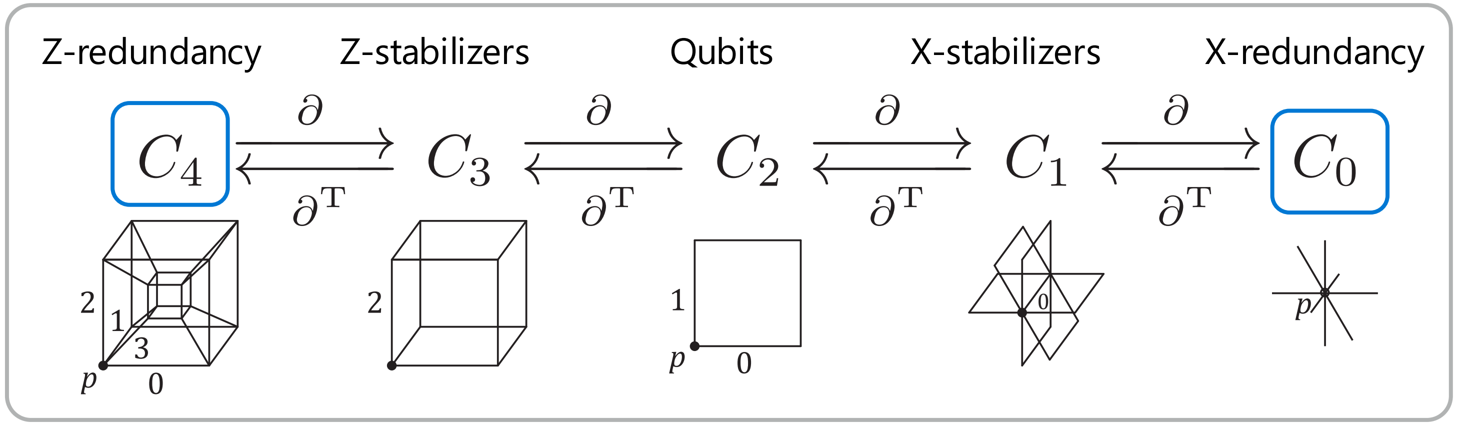


Eric Dennis, Alexei Kitaev, Andrew Landahl, and John Preskill. Topological quantum memory. Journal of Mathematical Physics, 43(9):4452–4505, 2002

5 term chain complex to enable single-shot property

Why 4D?

- Typical QEC codes use a 3-term chain complex: Z-stabilizers → Qubits → X-stabilizers
- a 5-term chain complex adds redundant stabilizers enabling single-shot error correction



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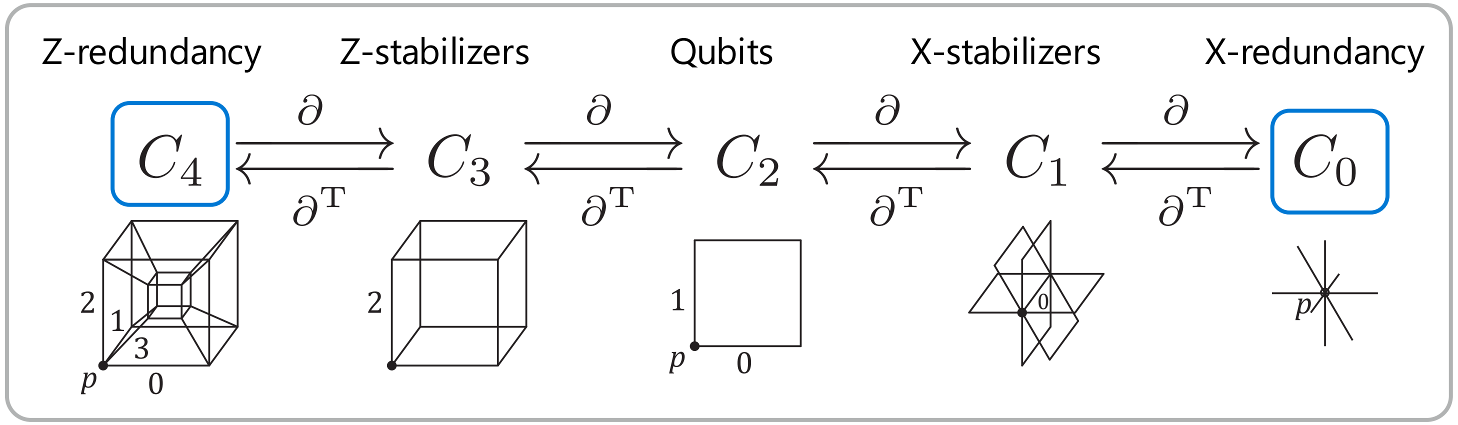

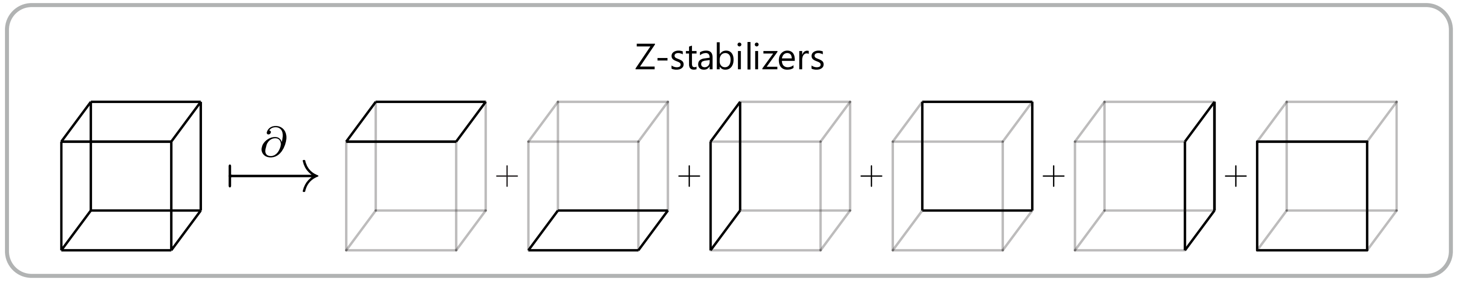

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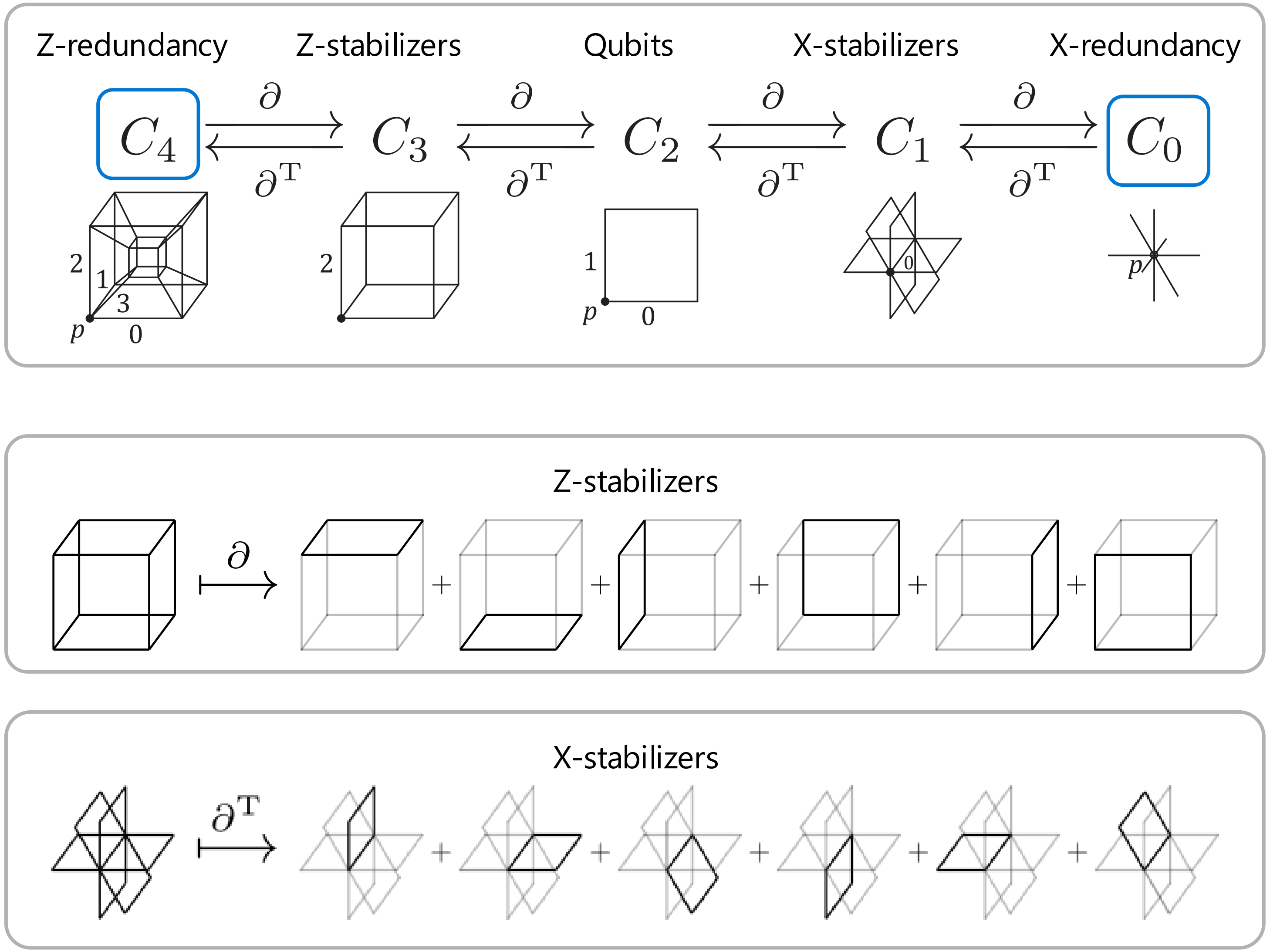


Single-shot decoding

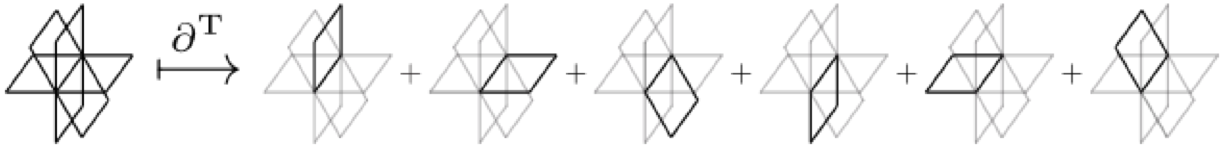
Reduces logical cycle time by ~d compared to traditional methods

- Stabilizer redundancies enforce syndrome patterns to form closed loops
- Measurement errors lead to open loops

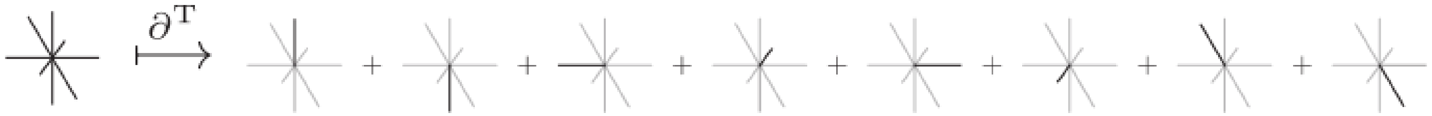
Single-shot decoding:

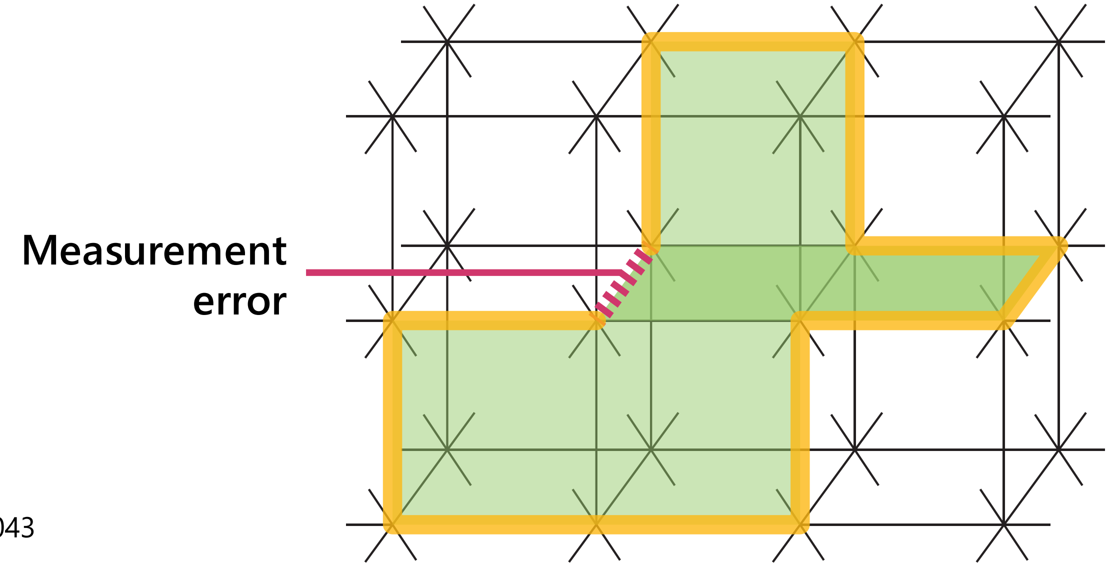
- Match endpoints to form closed loops
- Match loops with minimum weight membranes

X-stabilizers



X-redundancies





Bombin, Phys. Rev. X 5, 031043

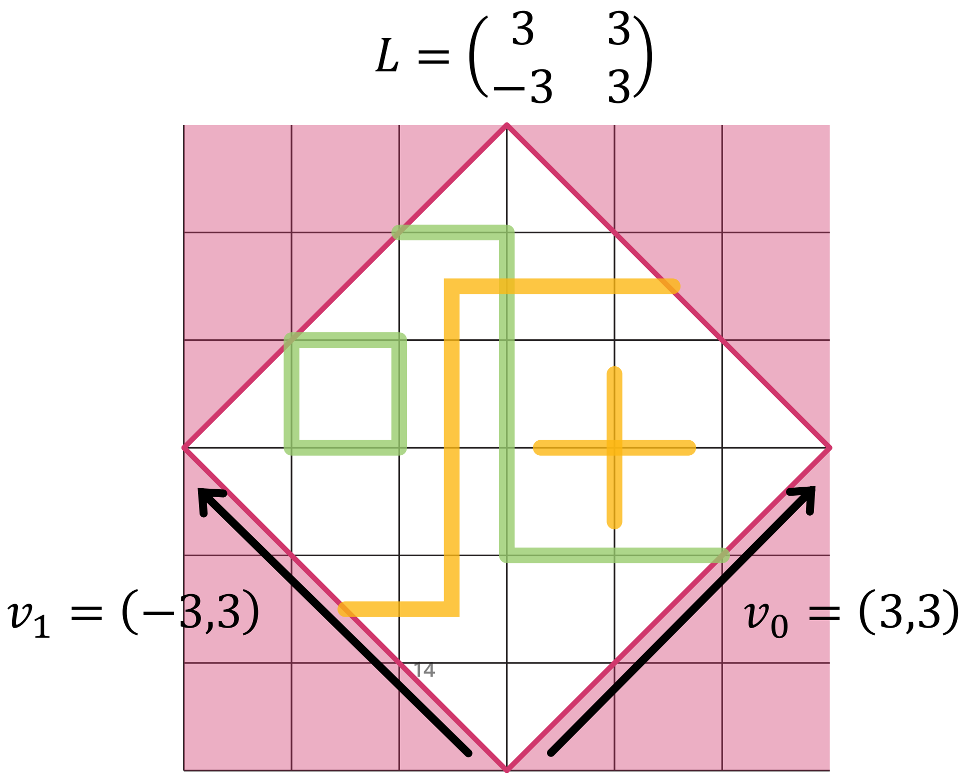
Rotated codes

General rotation is given by taking $\mathbb{Z}^2$ and quotienting by integer lattice

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

$$(x, y) \backsim (x, y) + v_0 \sim (x, y) + v_1$$

Different lattices give different cellulations of the torus



Rotated codes

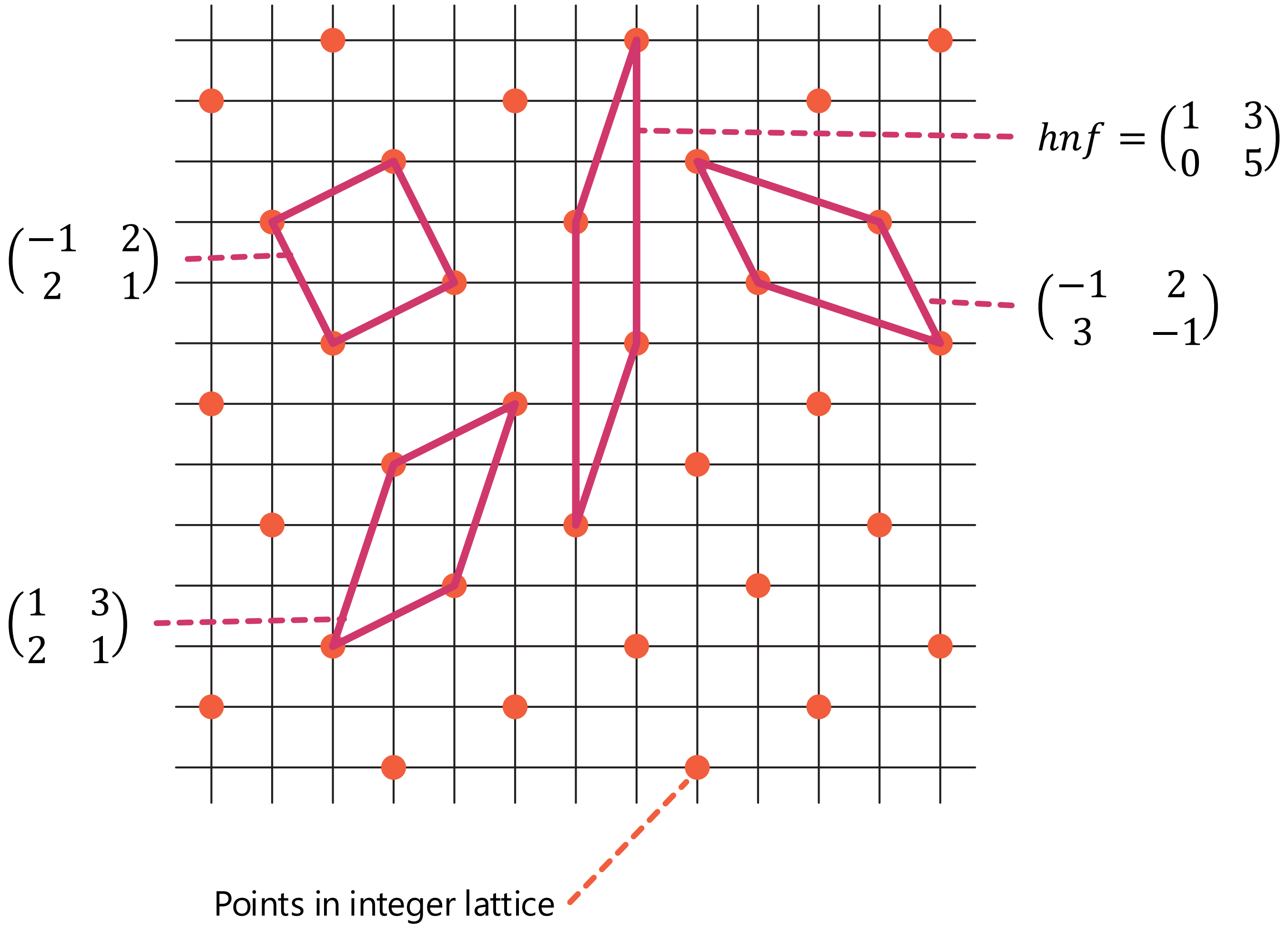
Equivalent lattices are generated by distinct basis vectors:

$$L = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}$$

Every integer lattice has a unique Hermite Normal Form representative:

$$L = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1_n} \\ 0 & a_{22} & \dots & a_{2_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

- Upper triangular $a_{ii} > 0, 0 \leq a_{ij} < a_{ii} (i < j)$
- Rows span the lattice



Code search

Counting

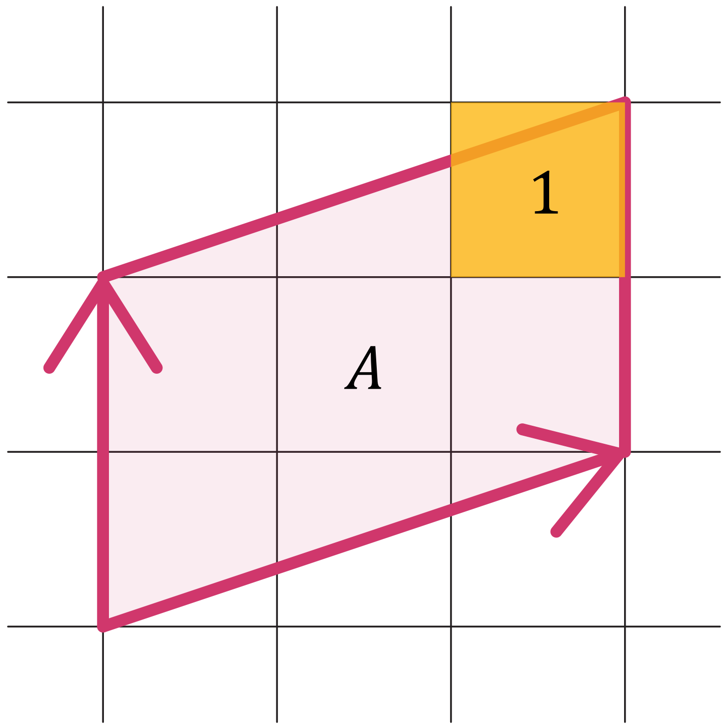
- 1 vertex per unit area
- (total area)/(unit area) = #vertices
- Total area = det(L)
- #qubits =2#vertices = 2 det(L)

Code distance is L^1 systole

$$d = \min_{c \text{ non-trivial cycle}} \ell^1(c)$$

Code Optimization:

- Enumerate Hermite normal forms for a fixed determinant
- Maximize ℓ^1 systole at fixed determinant to find optimal code parameters [[2det(L),2,d]]



Rotated loop only Toric code search

Each pair of directions forms a logical qubit

- 6 logical qubits: (0,1),(0,2),(0,3),(1,2),(1,3),(2,3)

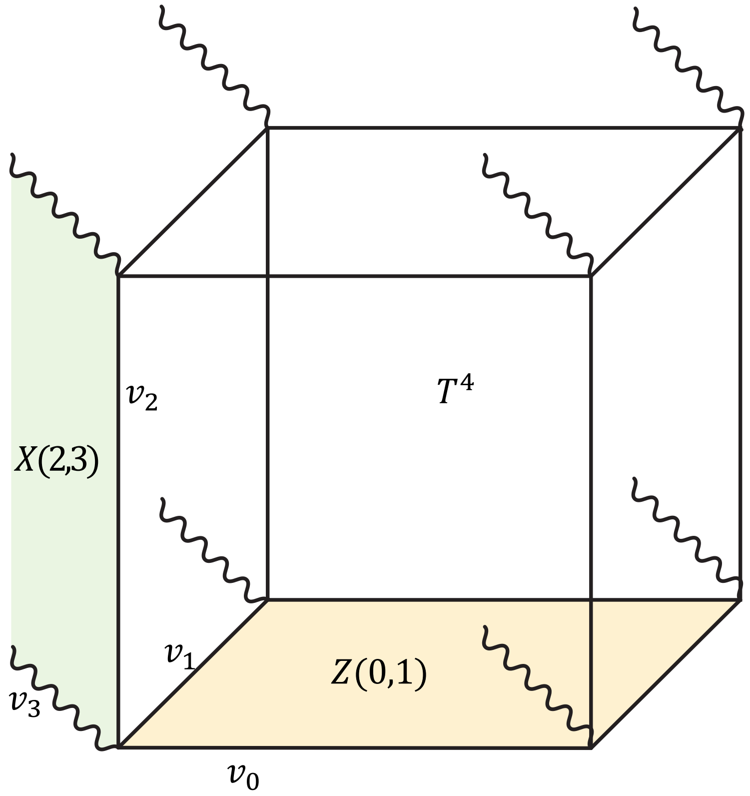
General "rotation" is given by taking $\mathbb{Z}^4$ and quotienting by integral lattice L

It is hard to compute code distance by brute force

- observation: all optimal codes also have optimal ℓ^1 systole
- ℓ^1 systole is much easier to calculate enabling an efficient code search

$$L = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Lattices are parametrized by Hermite Normal Form



Rotated loop only
Toric code search

$$\mathrm{det9}[[54,6,6]]:\begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&0&9\end{pmatrix}$$

$$Hadamard\ [[96,6,8]]:\begin{pmatrix}1&1&1&1\\0&2&0&2\\0&0&2&2\\0&0&0&4\end{pmatrix}$$

$$\mathrm{det45}\ [[270,6,15]]:\begin{pmatrix}1&0&1&6\\0&1&0&11\\0&0&3&9\\0&0&0&15\end{pmatrix}$$

Optimal lattice conjecture:

6x qubit reduction

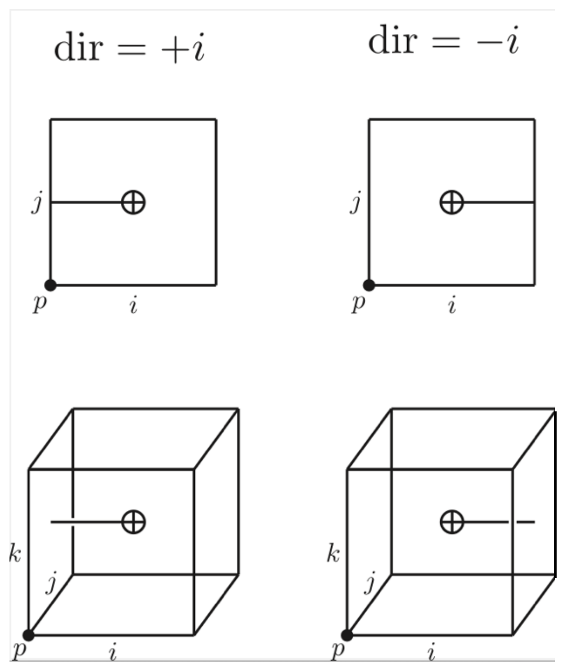
Det	n	k	d	d best pair
2	12	6	2	
3	18	6	3	
4	24	6	3	
5	30	6	4	
6	36	6	4	
7	42	6	4	5
8	48	6	4	
9	54	6	6	
15	90	6		
16	96	6	8	9
17	102	6	7	
18	108	6	9	
…				
45	270	6	15	

Det9 lattice → row Det 9
Hadamard lattice → row Det 16
Det45 lattice → row Det 45

L	Det	n	k	d	reduction
1	1	6	6	1	
2	16	96	6	4	3.2x
3	81	486	6	9	5.1x
4	256	1536	6	16	5.7x

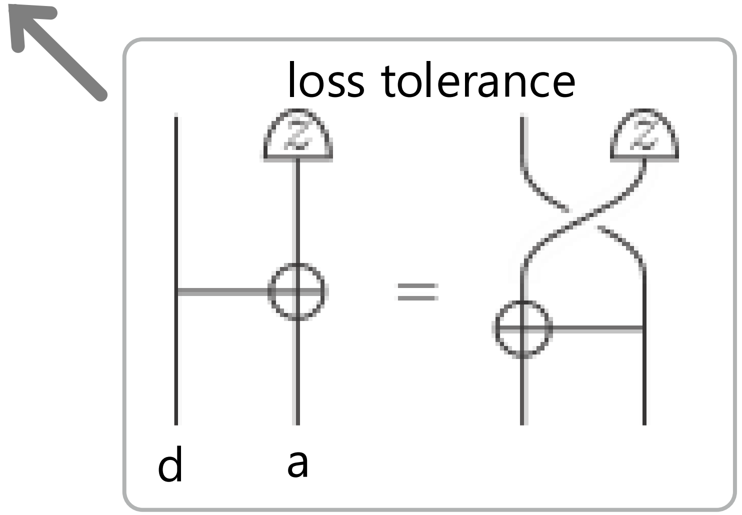
Syndrome extraction

Depth 8 syndrome extraction circuit



Circuit 2 Compact syndrome extraction circuit

```
for each a in 1-cells do
  PrepX(a)
end for
for each c in 3-cells do
  PrepZ(c)
end for
for each direction in {−3, −2, −1, −0, +0, +1, +2, +3} do
  for each a in 1-cells such that a + direction = b in 2-cells do
    CNOT(a, b)
  end for
  for each c in 3-cells such that c + direction = b in 2-cells do
    CNOT(b, c)
  end for
end for
for each a in 1-cells do
  MeasX(a)
end for
for each c in 3-cells do
  MeasZ(c)
end for
```



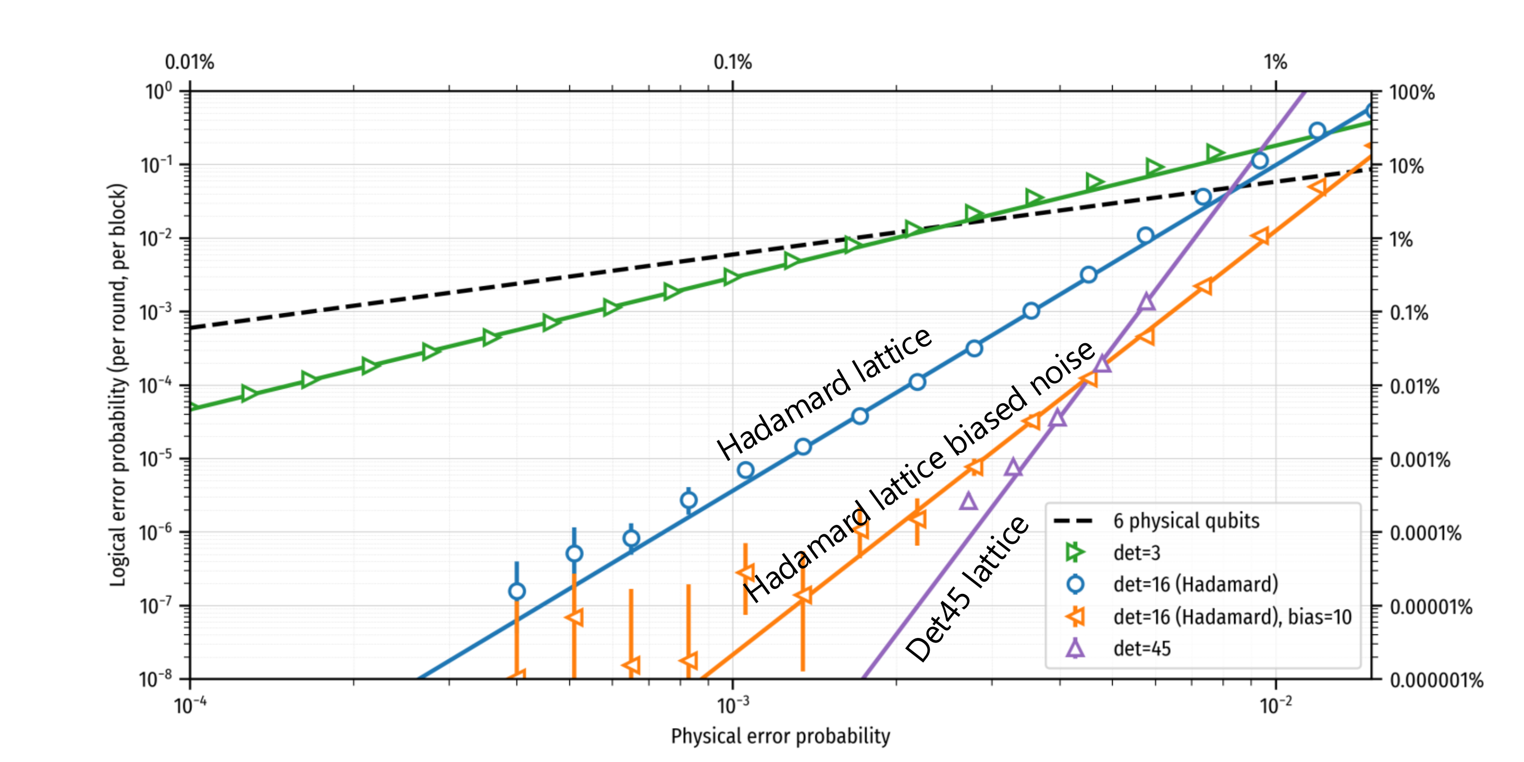
Performance simulations

Relay-BP decoding of (3,1) window size

relayBP: Muller et al https://arxiv.org/abs/2506.01779

Noise model

Standard circuit level noise



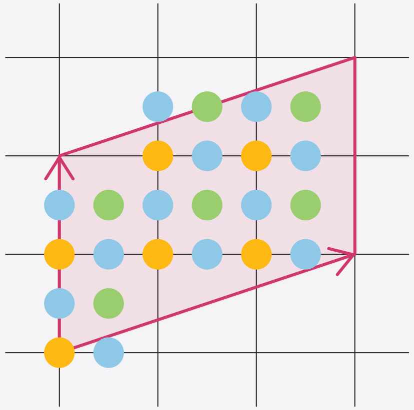
Codes for comparison	Code parameters	PL(0.001)/k	PL(0.001)	Pseudo-threshold
4D geometric code	[[96,6,8]]	4×10^{-7}	2×10^{-6}	0.01
Rotated surface code (arXiv:2409.14765)	[[64,1,8]]	7×10^{-6}	7×10^{-6}	0.0024
BB code (Nature, 627(8005):778–782)	[[90,8,10]]	6×10^{-7}	5×10^{-6}	0.0053
SHYPS (arXiv:2502.07150)	[[225,16,8]]	3×10^{-5}	5×10^{-4}	0.003

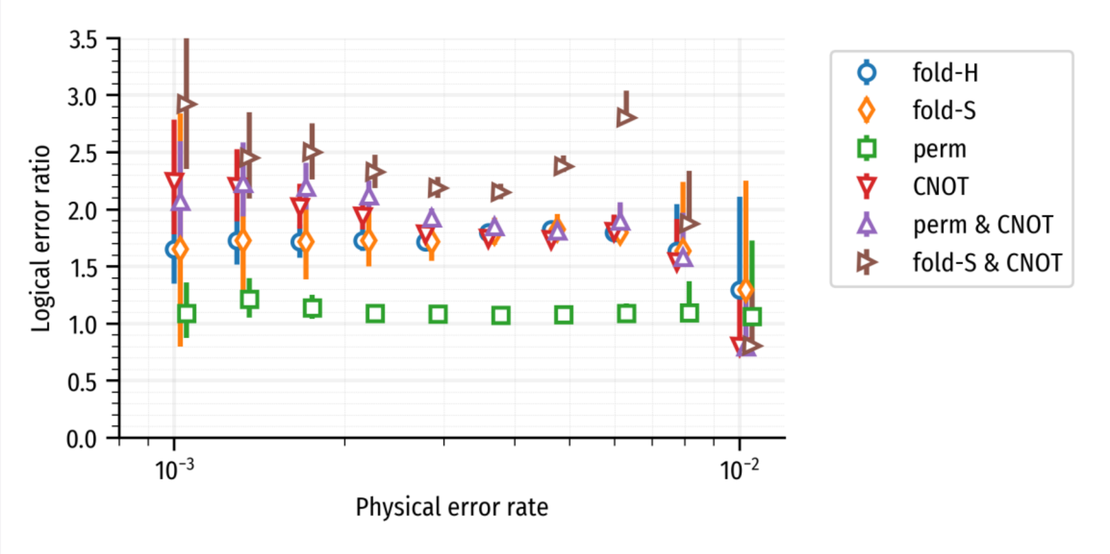
Logical operations

(1) Transversal operations from 4D crystal:

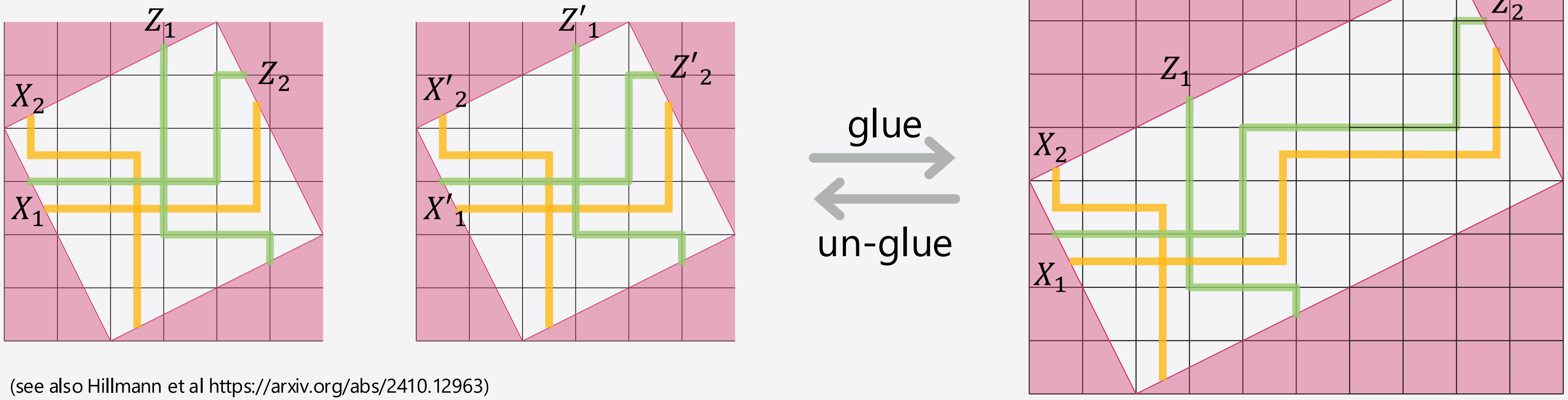
- Permutation
- Fold-H
- Fold-S

Breuckmann, Burton Quantum 8, 1372 (2024)





(2) Lattice surgery



(see also Hillmann et al https://arxiv.org/abs/2410.12963)

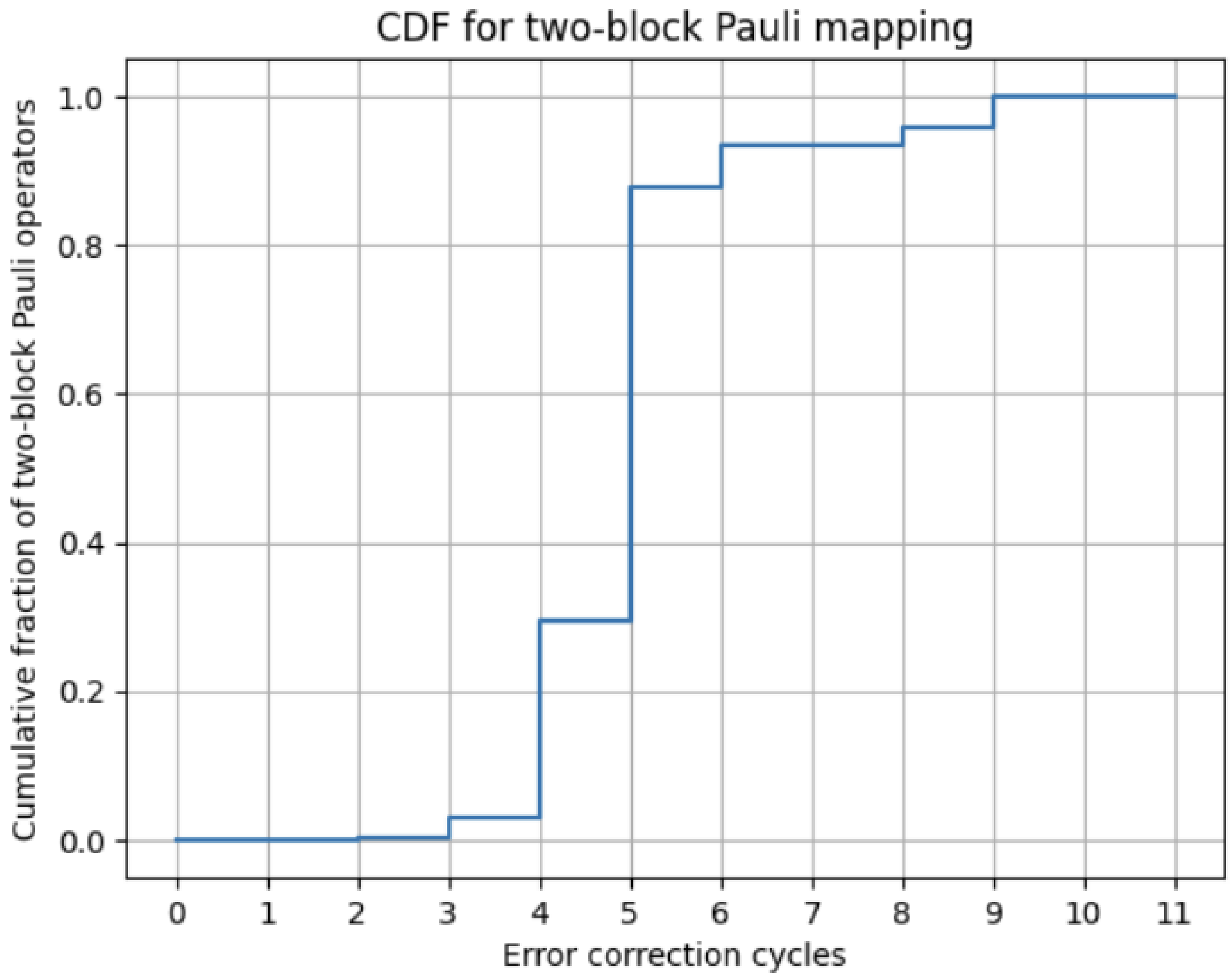
Clifford Synthesis

Clifford gates are combined with lattice surgery to form a Clifford complete gate set

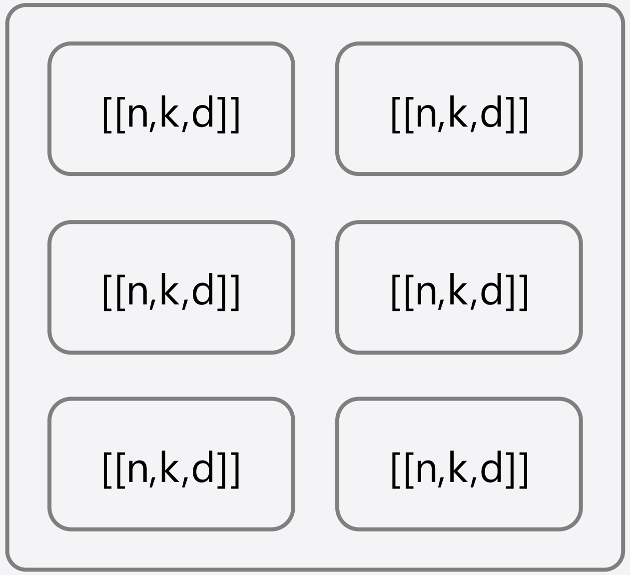
Overhead:

- One logical ancilla block as a resource state
- Plotted is the number of error correction cycles to map

$$Z_1 \longrightarrow P_1 P_2 \ldots P_{12}$$
$$P_j \in \{I, X, Y, Z\}$$



Outlook



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- ☑ Minimize logical failure rate → high code distance
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- ☑ Maximize computational spacetime volume → efficient compilation of logical operations

Enabling non-local gates unlocks a vast optimization space for hardware/QEC co-design