Fault-tolerant quantum input/output

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M. Christandl, A. Goswami and O. Fawzi, Fault-tolerant quantum computation with constant overhead for general noise, PRX Quantum 6, 040334, 2025

M. Christandl, A. Goswami and O. Fawzi, Fault-tolerant quantum input/output https://arxiv.org/abs/2408.05260 QIP'25

Paula Belzig, M. Christandl and A. Müller-Hermes, Fault-tolerant coding for entanglement-assisted quantum communication, *IEEE Transactions in Information Theory* **70**, 2655 (2024)

M. Christandl and A. Müller-Hermes, Fault-tolerant coding for quantum communication, *IEEE Transactions on Information Theory*, **70**, 282 (2024)

Overview

- Fault-tolerant Quantum Computing
- Fault-tolerant Quantum I/O
- Applications:
  - Fault tolerant Quantum Communication
  - Fault-tolerant Quantum Computation with Constant Overhead for General Noise

Fault-tolerant Quantum Computing

Quantum algorithms

Assumption of noiseless quantum computer is unrealistic!

- factoring

15 → Shor → 5*3

781 → Shor → 71*11

⋮

6303703466840400359 → Shor → 1055539229*5972021971

- search

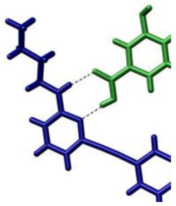
+45 51824325 → Grover → Matthias Christandl

+33 4 37 37 60 00 → Grover → ENS Lyon

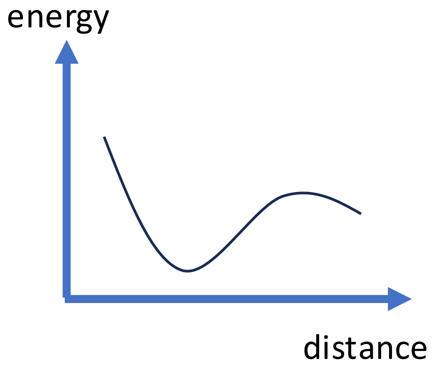
⋮

+1 617 84729778291 → Grover → Peter Shor?

- chemistry



→ Quantum Phase Estimation →



Quantum algorithms

Assumption of noiseless quantum computer is unrealistic!

- factoring

15 → Shor → 7*3?

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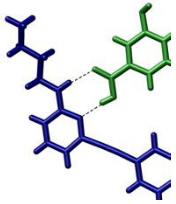
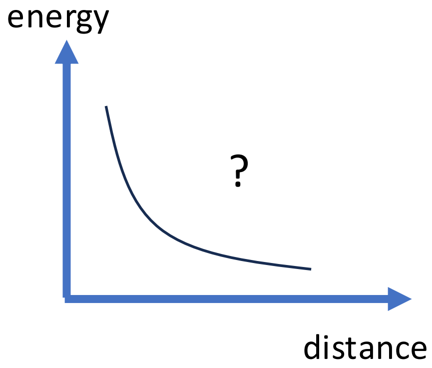


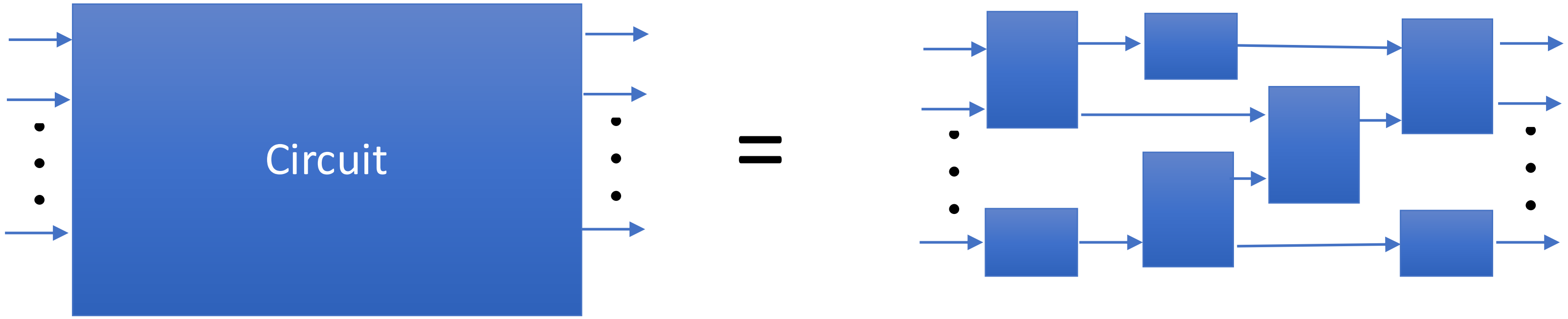
→ Quantum Phase Estimation →



How can we combat the noise?

Quantum circuit

- pick a universal gate set (X, Y, Z, T, CNOT, id, prepare 0, 1, partial trace)



- qubits are implemented in
  - ions, atoms (gates=laser pulses)
  - superconducting qubits, spin qubits
    (gates are microwave pulses)
  - photonic qubits
    (beam splitters, phase shifters, non-linear elements)

Each gate has
error of 10^{-2} to 10^{-4}

decoherence
 - interaction with environment
 - amplitude damping, dephasing
imprecise implementation
 - overrotation
 - leakage

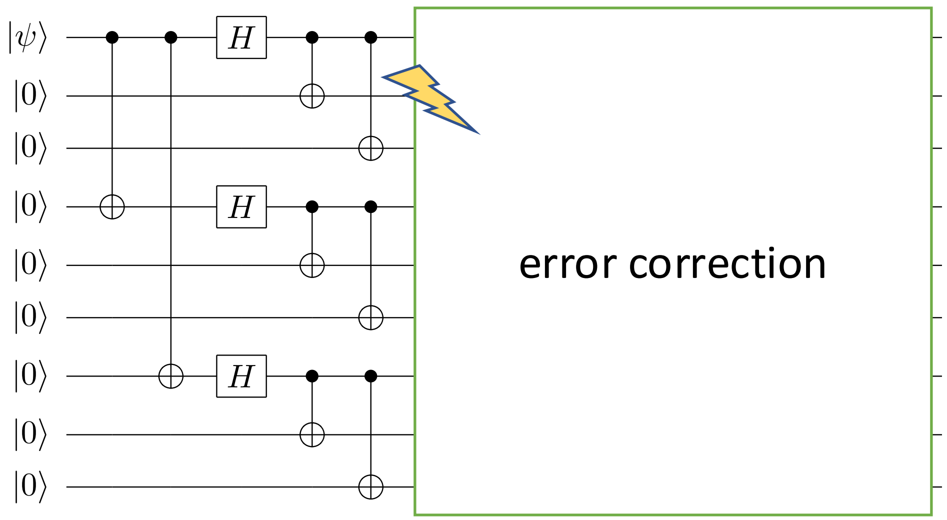
Error correction

- Error correction (repetition code)
  - 0→(encoding) 000 → (1 error) 010 → (decoding) 0
  - 1→(encoding) 111 → (1 error) 101 → (decoding) 1

- Quantum error correction (Shor code)

$$|0_S\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

$$|1_S\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$



- Error correction works

source wiki

Error correction

- Error correction (repetition code)
  - 0→(encoding) 000 → (1 error) 010 → (decoding) 0
  - 1→(encoding) 111 → (1 error) 101 → (decoding) 1

- Quantum error correction (Shor code)

$$|0_S\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

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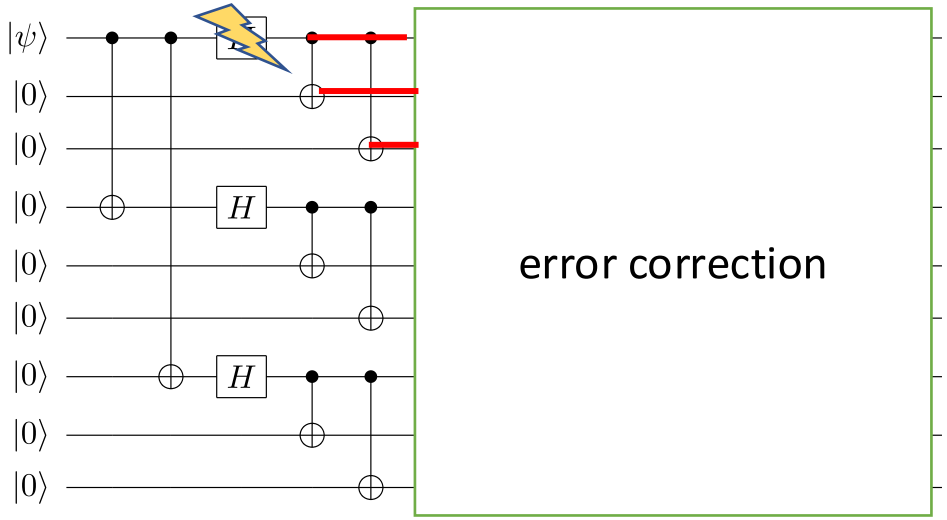
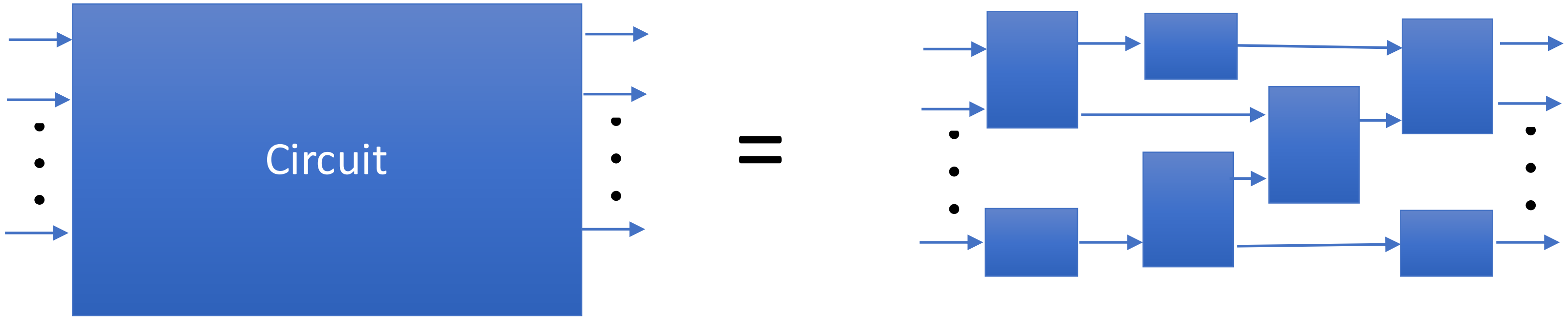


- Error spreads→error correction fails

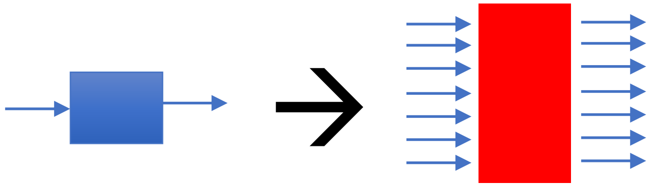
source wiki

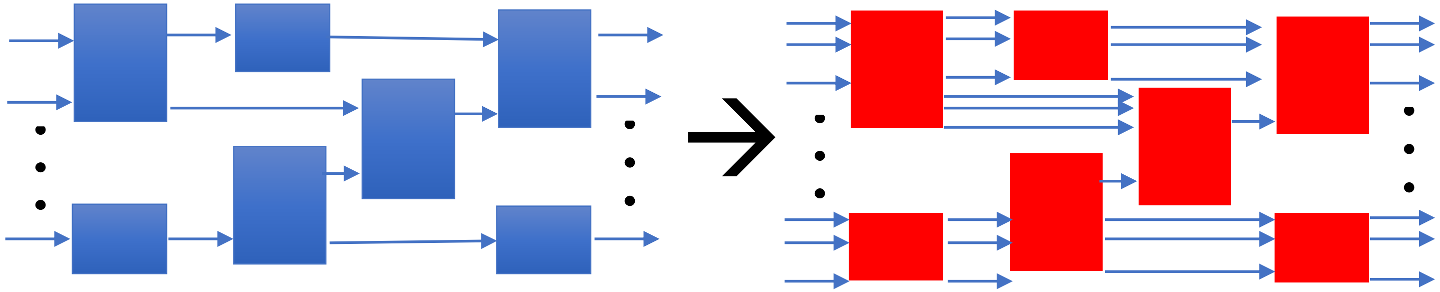
Fault-Tolerant Constructions

- pick a universal gate set (X, Y, Z, T, CNOT, id, prepare 0, 1, partial trace)

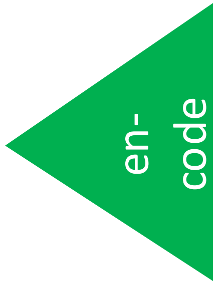


- every gate will be replaced by a gadget





Encoding



from M. Grassl

Bounds on [[11,1]]$_2$

lower bound: 5
upper bound: 5

Construction

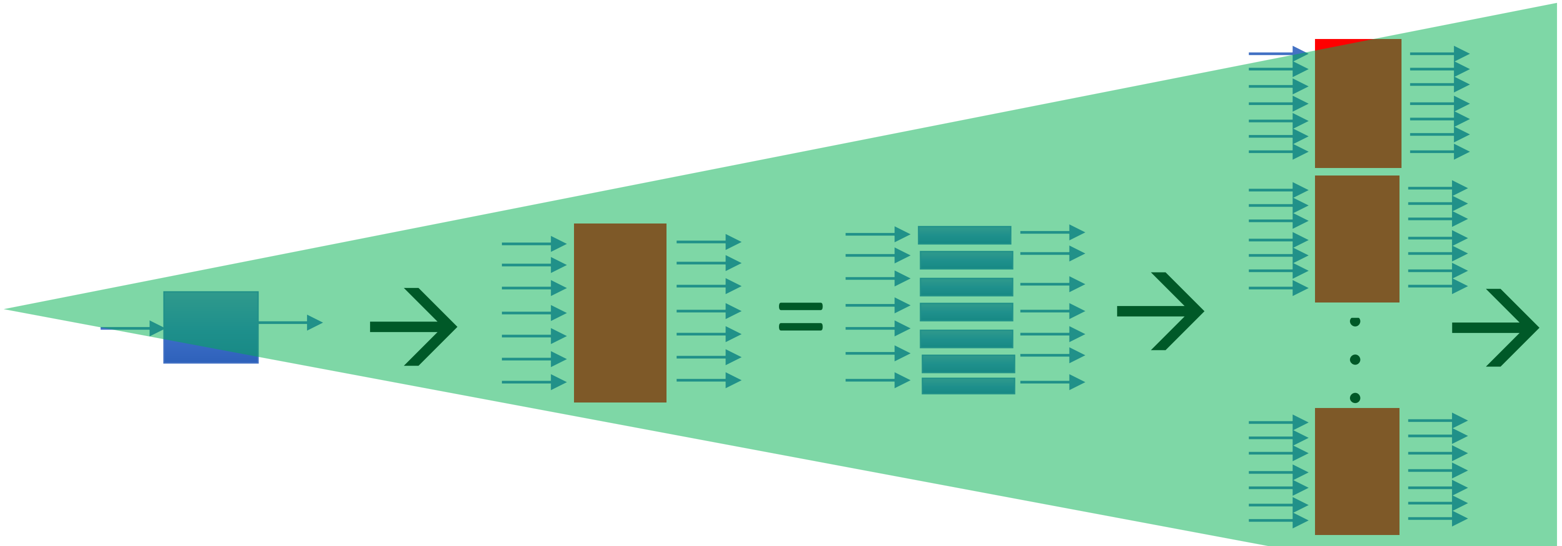
```
Construction of a [[11,1,5]] quantum code:
[1]:  [[11, 1, 5]] quantum code over GF(2^2)
     Construction from a stored generator matrix

    stabilizer matrix:

      [1 0 0 0 0 1 0 0 0 1 1|0 0 0 0 0 0 1 1 0 0 0]
      [0 0 0 0 0 0 1 1 1 0 1|1 0 0 0 0 1 0 1 1 0 0]
      [0 1 0 0 0 1 0 1 0 0 1|0 0 0 0 0 0 1 1 1 0 1]
      [0 0 0 0 0 0 0 0 1 1 0|0 1 0 0 0 1 1 1 1 1 0]
      [0 0 1 0 0 1 0 1 1 0 0|0 0 0 0 0 0 1 0 0 1 0]
      [0 0 0 0 0 0 1 0 1 0 0|0 0 1 0 0 1 1 0 1 1 1]
      [0 0 0 1 0 1 1 0 1 1 1|0 0 0 0 0 0 1 0 0 0 1]
      [0 0 0 0 0 0 0 0 1 0 1|0 0 0 1 0 1 0 1 0 1 0]
      [0 0 0 0 1 1 1 0 0 0 1|0 0 0 0 0 0 0 0 1 1 0]
      [0 0 0 0 0 0 0 1 0 1 0|0 0 0 0 1 1 0 1 1 1 1]
```

- We mostly use codes that correct two or more errors (distance larger than 5)

- Concatenate for better error supression



#qubits increases exponentially

gates improve doubly exponentially

[11, 1, 5] quantum code

Stabilizer generators: Commuting Pauli operators generating a group

$$G_1 = Z\ Z\ Z\ Z\ Z\ Z\ I\ I\ I\ I\ I$$
$$G_2 = X\ X\ X\ X\ X\ X\ I\ I\ I\ I\ I$$
$$G_3 = I\ I\ I\ Z\ X\ Y\ Y\ Y\ Y\ X\ Z$$
$$G_4 = I\ I\ I\ X\ Y\ Z\ Z\ Z\ Z\ Y\ X$$
$$G_5 = Z\ Y\ X\ I\ I\ I\ Z\ Y\ X\ I\ I$$

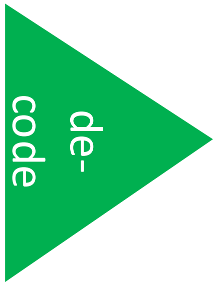
$$G_6 = X\ Z\ Y\ I\ I\ I\ X\ Z\ Y\ I\ I$$
$$G_7 = I\ I\ I\ Z\ Y\ X\ X\ Y\ Z\ I\ I$$
$$G_8 = I\ I\ I\ X\ Z\ Y\ Z\ X\ Y\ I\ I$$
$$G_9 = Z\ X\ Y\ I\ I\ I\ Z\ Z\ Z\ X\ Y$$
$$G_{10} = Y\ Z\ X\ I\ I\ I\ Y\ Y\ Y\ Z\ X$$

Stabilizer group: $S = \langle G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10} \rangle$

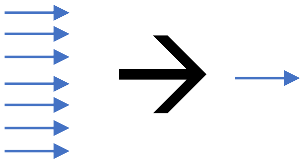
Logical states: simultaneous eigenstates with eigenvalue +1 of operators in S. For example,

$$|0\rangle_L \propto \sum_{s \in S} S|0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\rangle$$

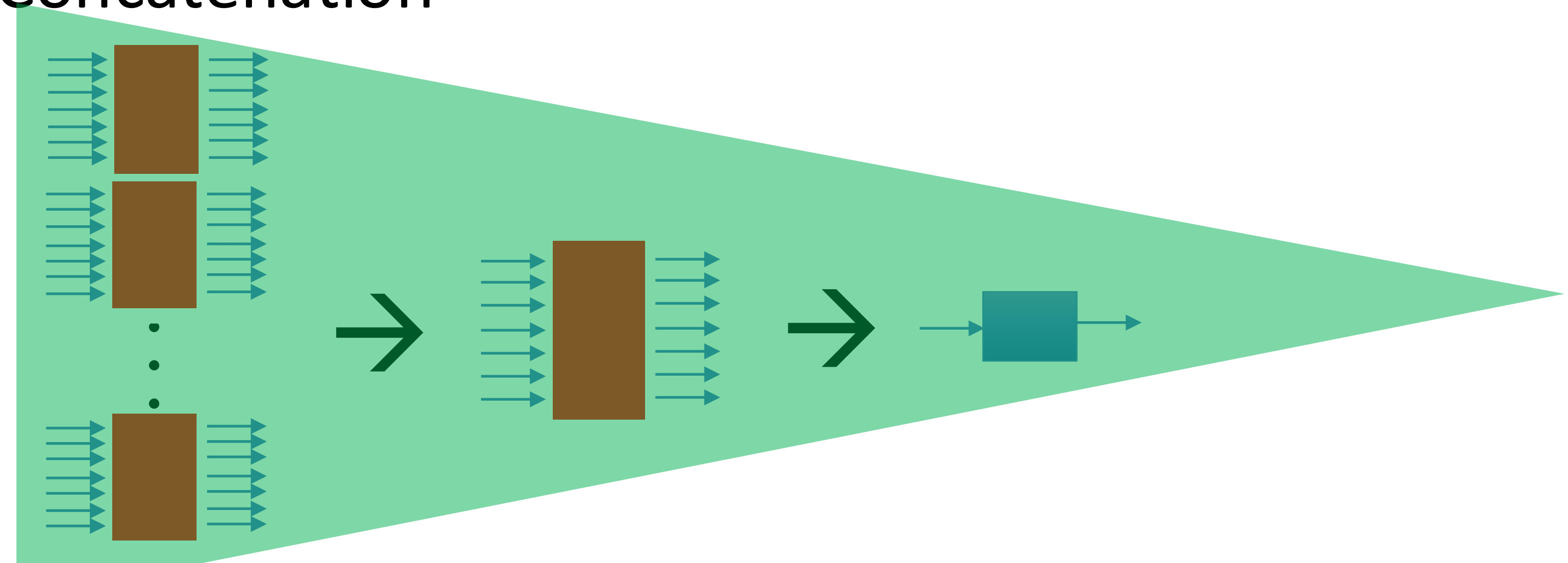
Decoding



- Error correction and decoding

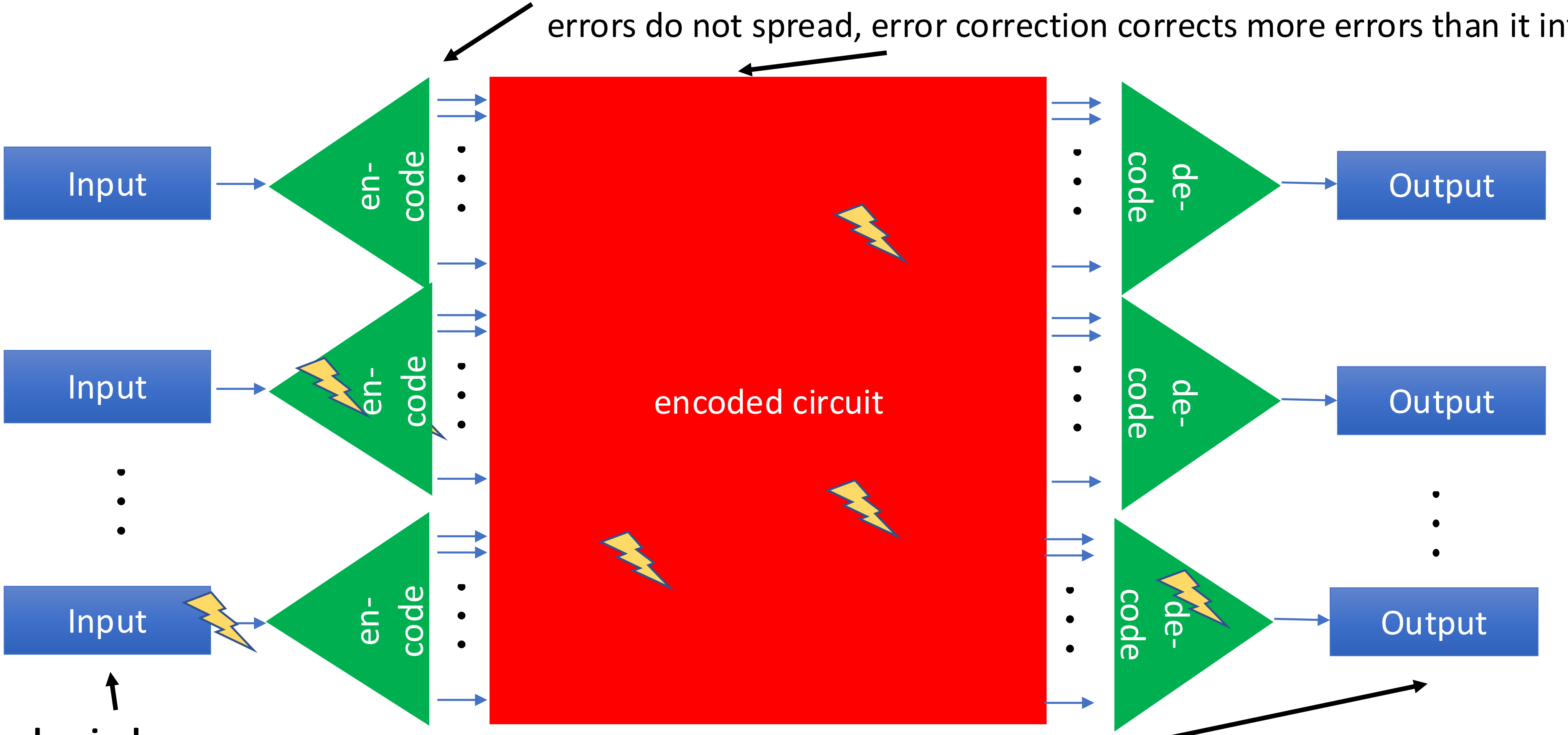


- Concatenation



Fault-tolerant quantum computing

special construction
errors do not spread, error correction corrects more errors than it introduces



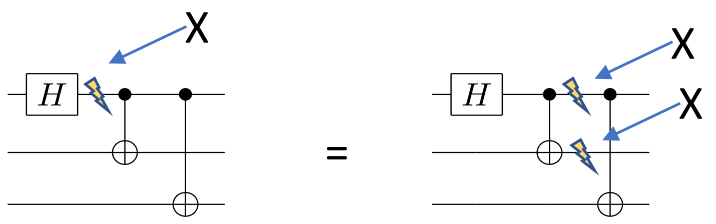
classical
assumed to be stable or already in error correcting code

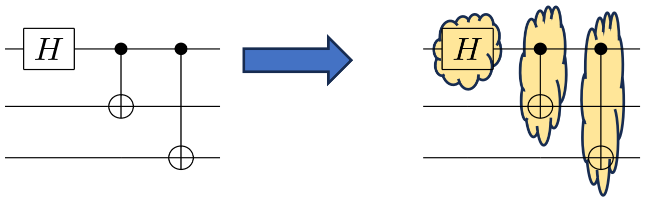
Error models

- Nice to think of errors happening probabilistically
  - Say, after each gate, each wire is affected by Pauli X, Y or Z error with probability δ (Pauli noise)
  - or some other map happens with probability δ (stochastic noise)
  - Note, if noise is i.i.d. on the locations, it still gets correlated on the physical qubits
  - Analysis via error patterns (cf. Gottesman et al.)
- More realistic
  - Each gate is replaced by some channel that is δ-close in diamond norm
  - Models the above, but includes also coherent rotations, amplitude damping, etc.
  - Analysis in Kitaev's framework (diamond norm)

$$\rho \mapsto (1-\delta)\rho + \frac{\delta}{3}(\sigma_X \rho \sigma_X + \sigma_Y \rho \sigma_Y + \sigma_Z \rho \sigma_Z)$$

$$\rho \mapsto (1-\delta)\rho + \delta\Lambda(\rho)$$





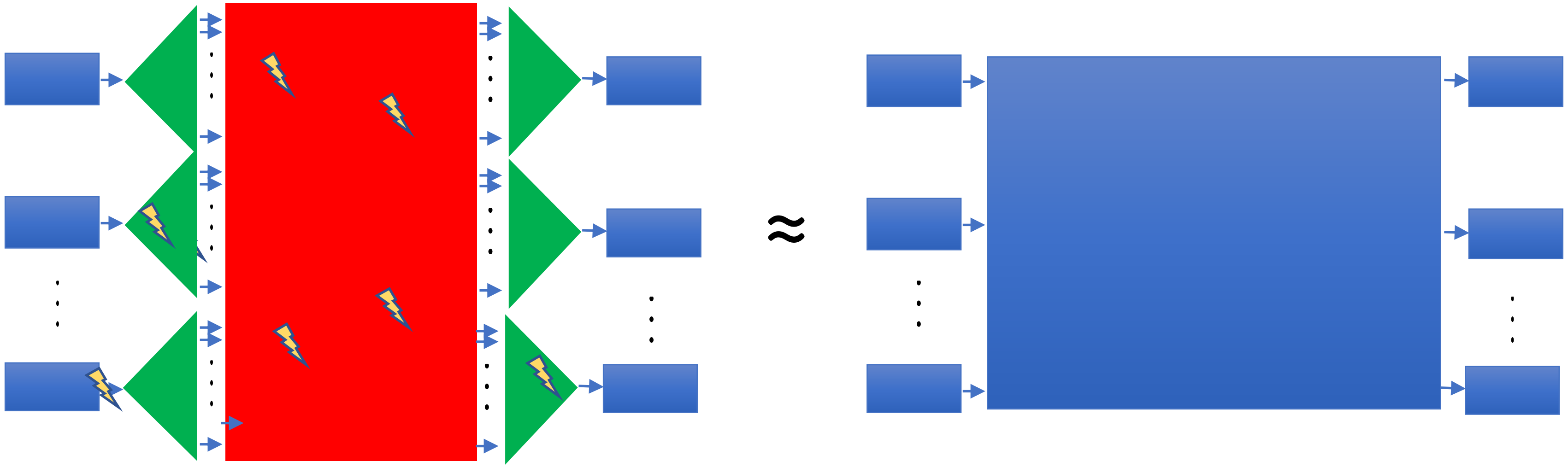
Threshold theorem for Quantum Computation

(à la Aliferis, Gottesman, Preskill, 2006)

- There is a threshold $\delta_0 > 0$ s.th. for all $\delta \leq \delta_0$ and all circuits and inputs

Prob[Output(k-level, δ -noise)≠Output (0-level, 0-noise]<O(#gates) $\left(\frac{\delta}{\delta_0}\right)^{2^k}$

stochastic

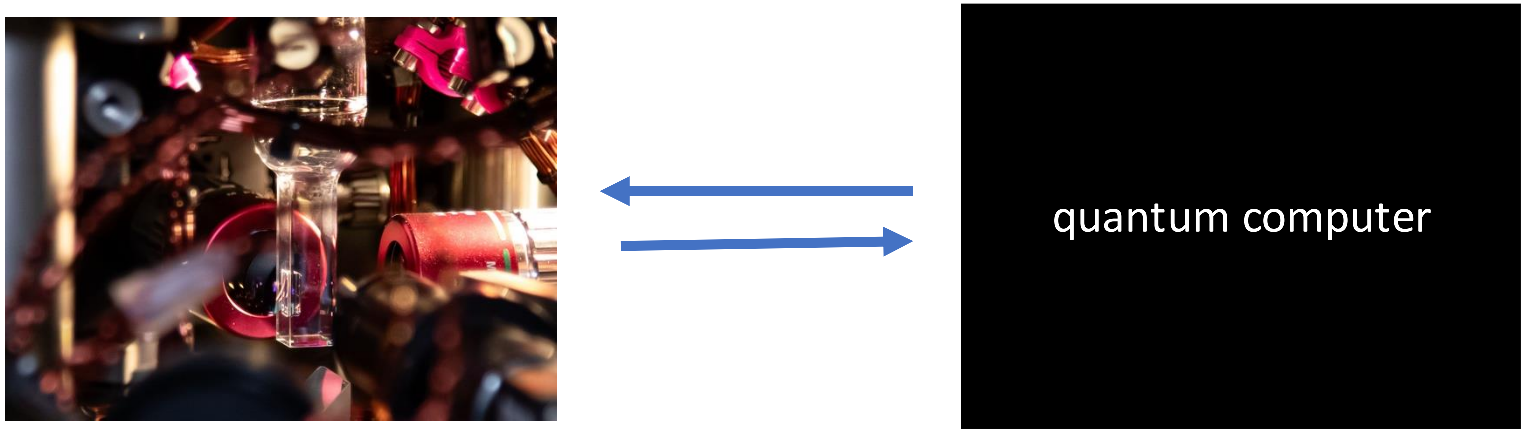


Classical input/output

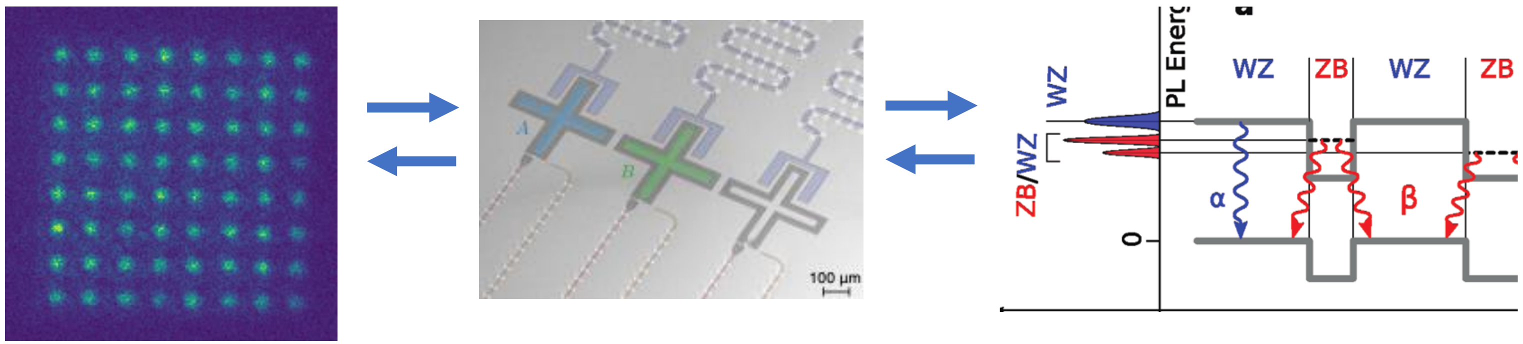
Quantum I/O

What if….

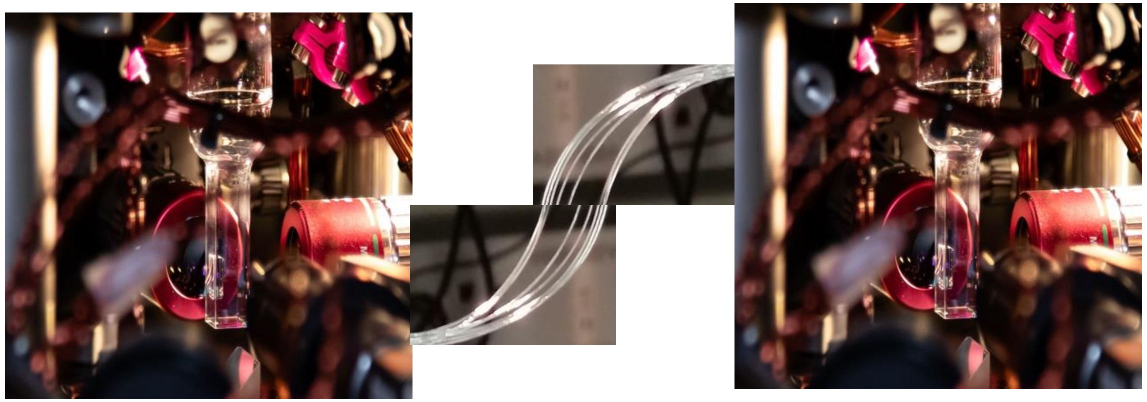
- … our noisy device interacts with a quantum black box?



- … we need to distribute the computation to computers based on different fault-tolerant architectures



- … noisy devices need to communicate via a realistic communication line?



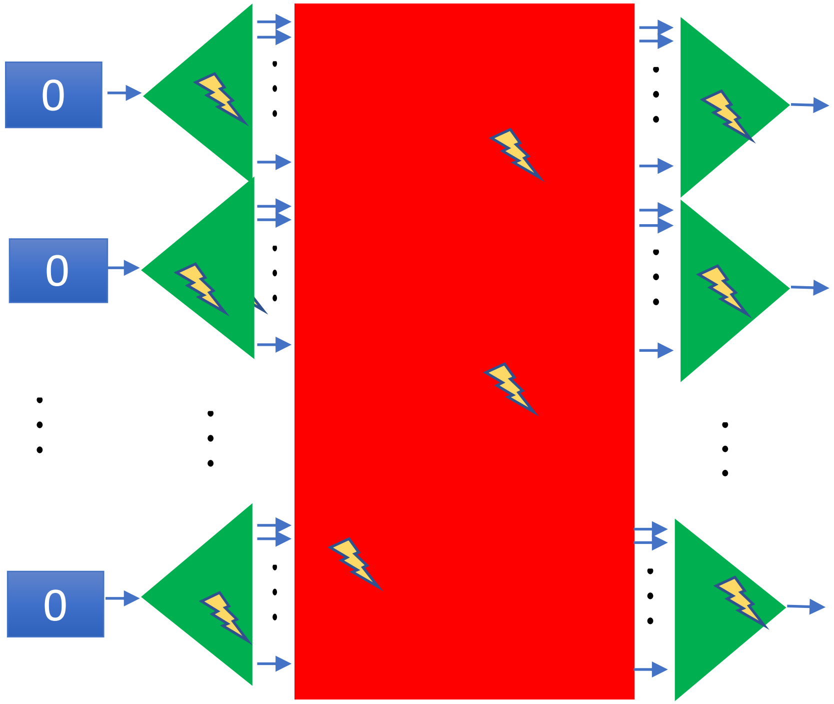
What if...

- ... we need to receive quantum information
  - ask for it delivered in our favorite quantum error correcting code?
  - puts burden on the sender
- ... we need to send quantum information
  - just send it in our favorite quantum error correcting code?
  - puts burden on the receiver
- Someone needs to solve the problem of

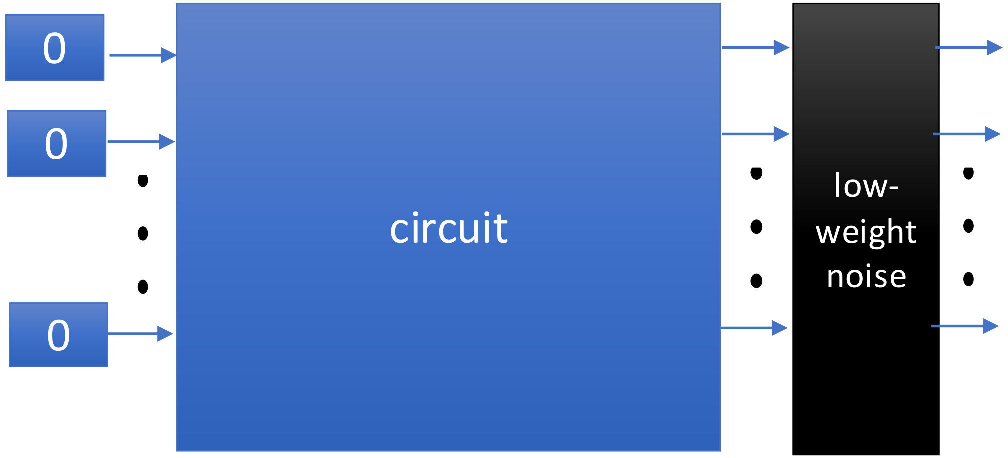
quantum input/output

State preparation for general noise

Theorem (Ch., Fawzi, Goswami)



≈

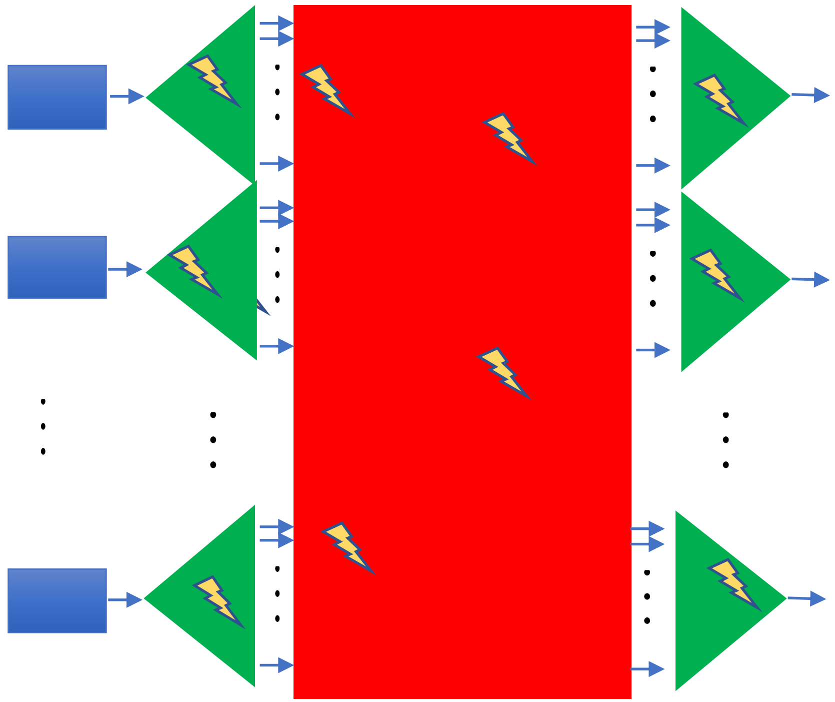


quantum input/output

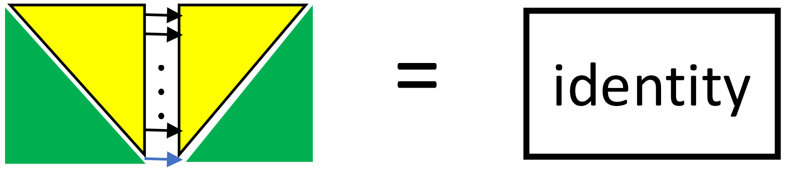
- optimal, as errors can happen in the end
- in two noise models
  - general noise model incl coherent noise
  - stochastic noise (previous work by Gottesman)

- similar statement for receiving quantum information
- also statements for quantum input & output

Proof



- insert ideal decoder/encoder pair separating logical qubit from syndrome



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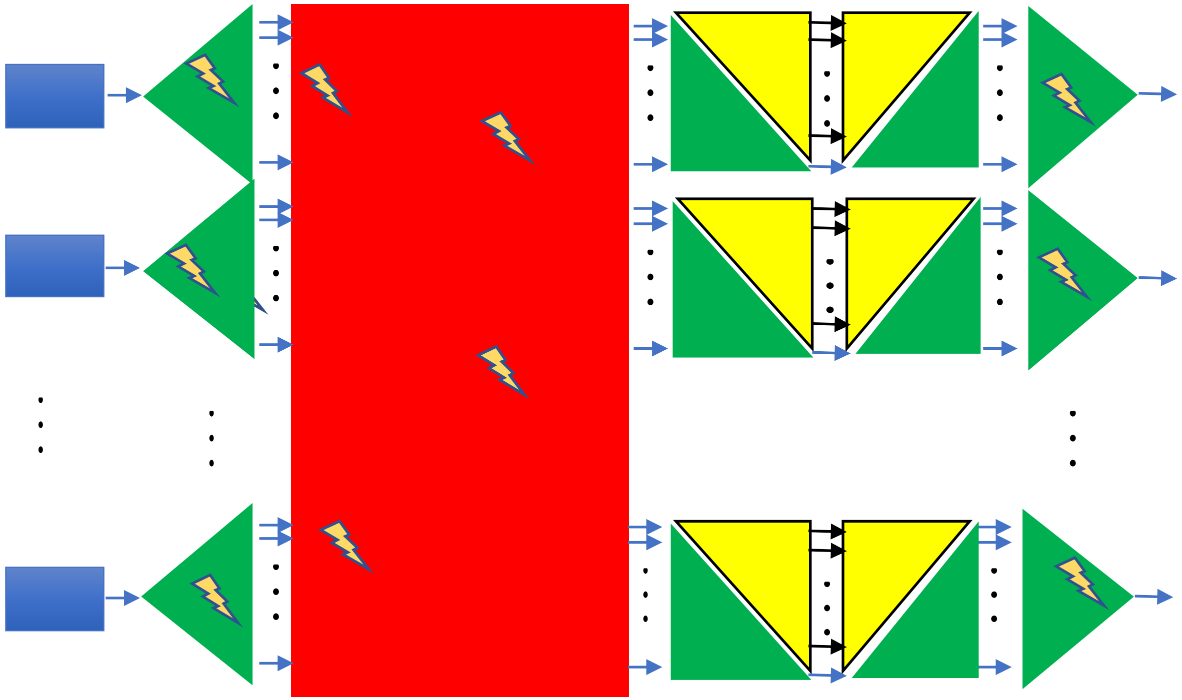

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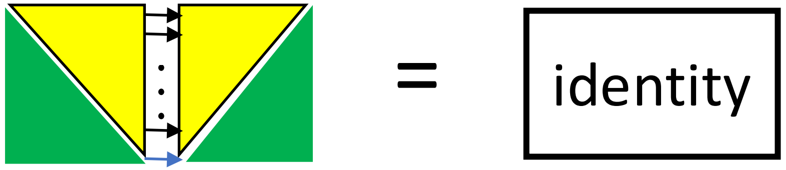

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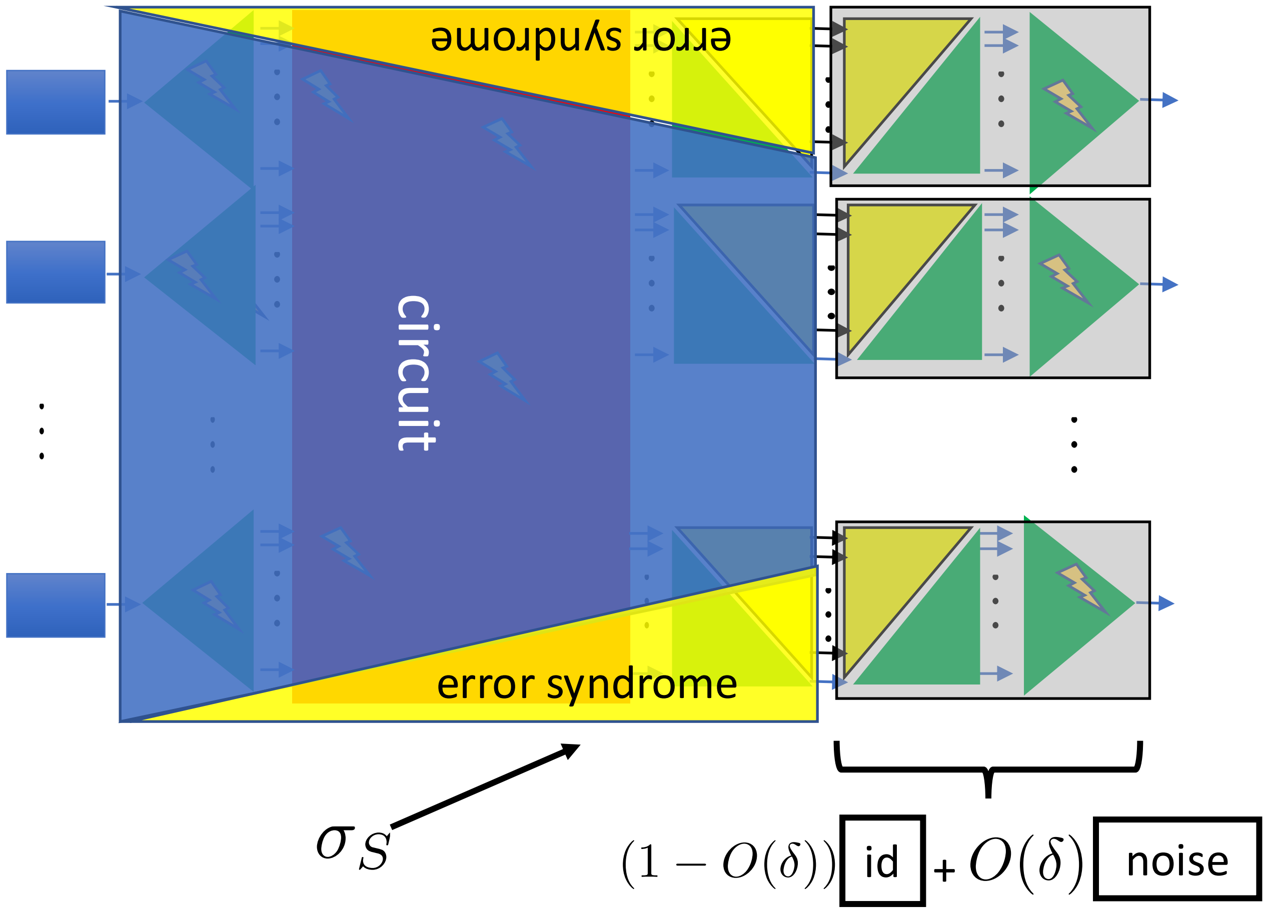
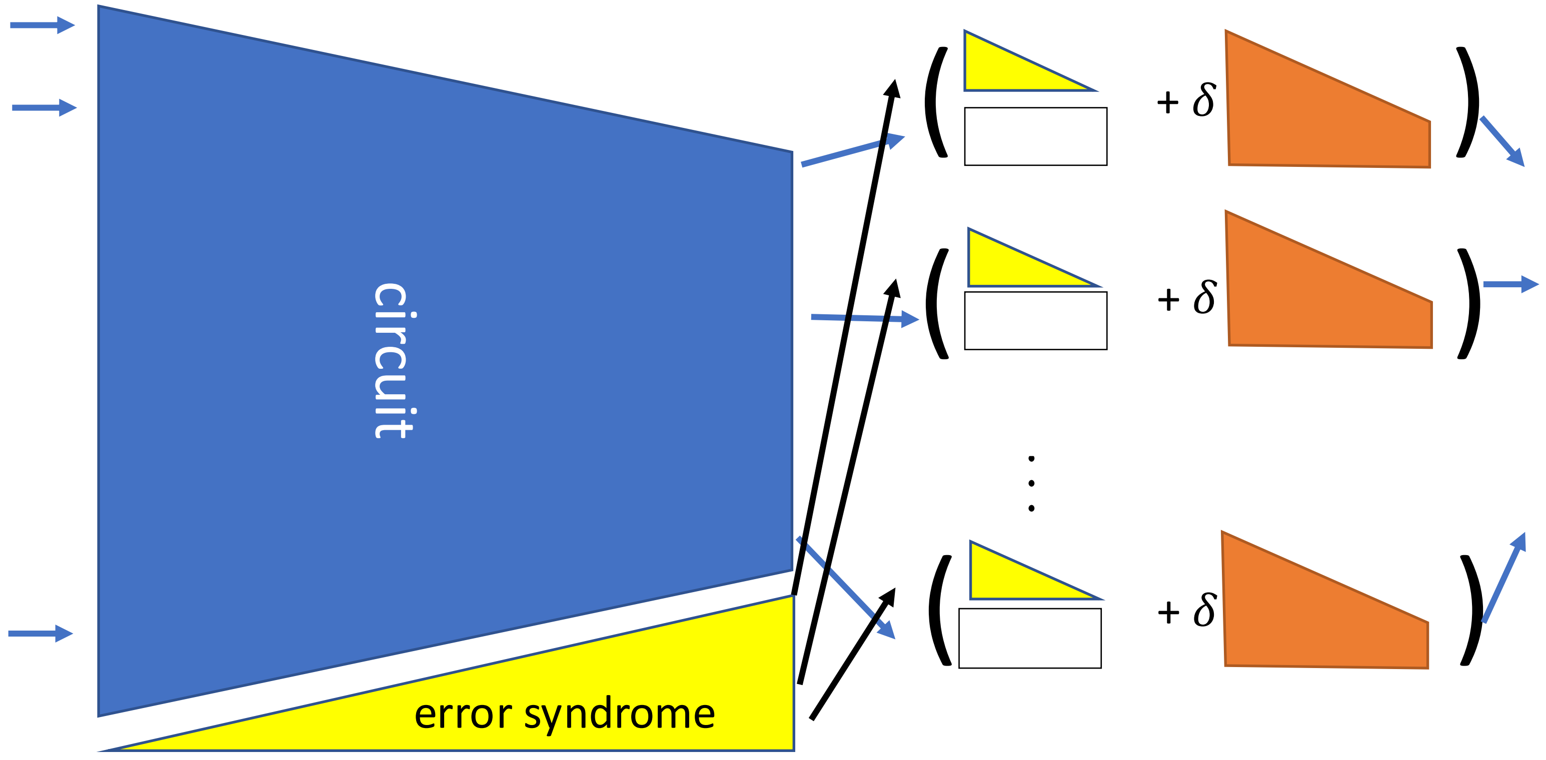


decoupled from input
correlated across qubits

Proof



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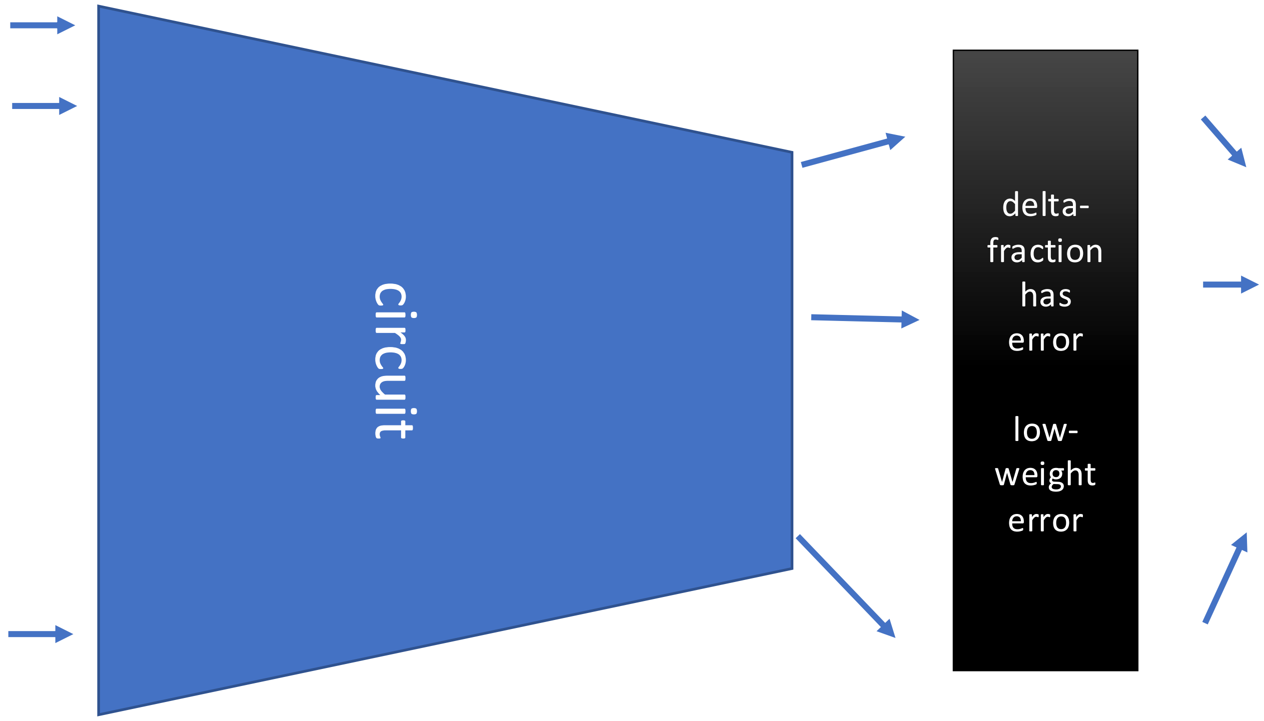
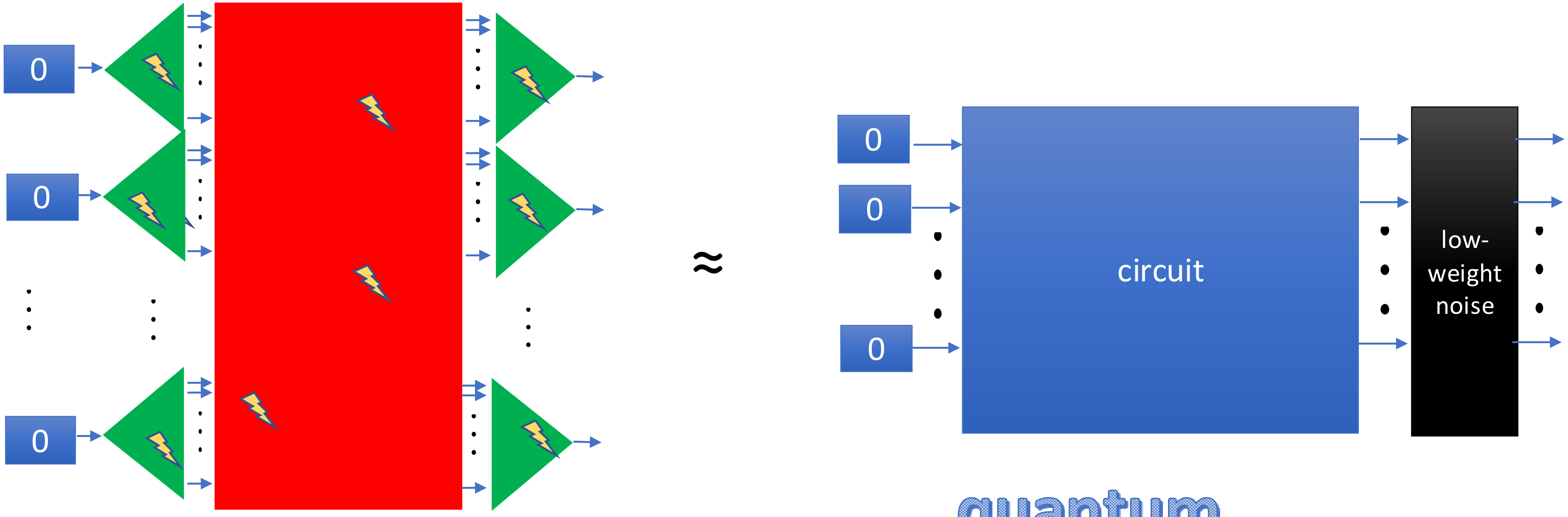


State preparation for general noise

Theorem (Ch., Fawzi, Goswami)



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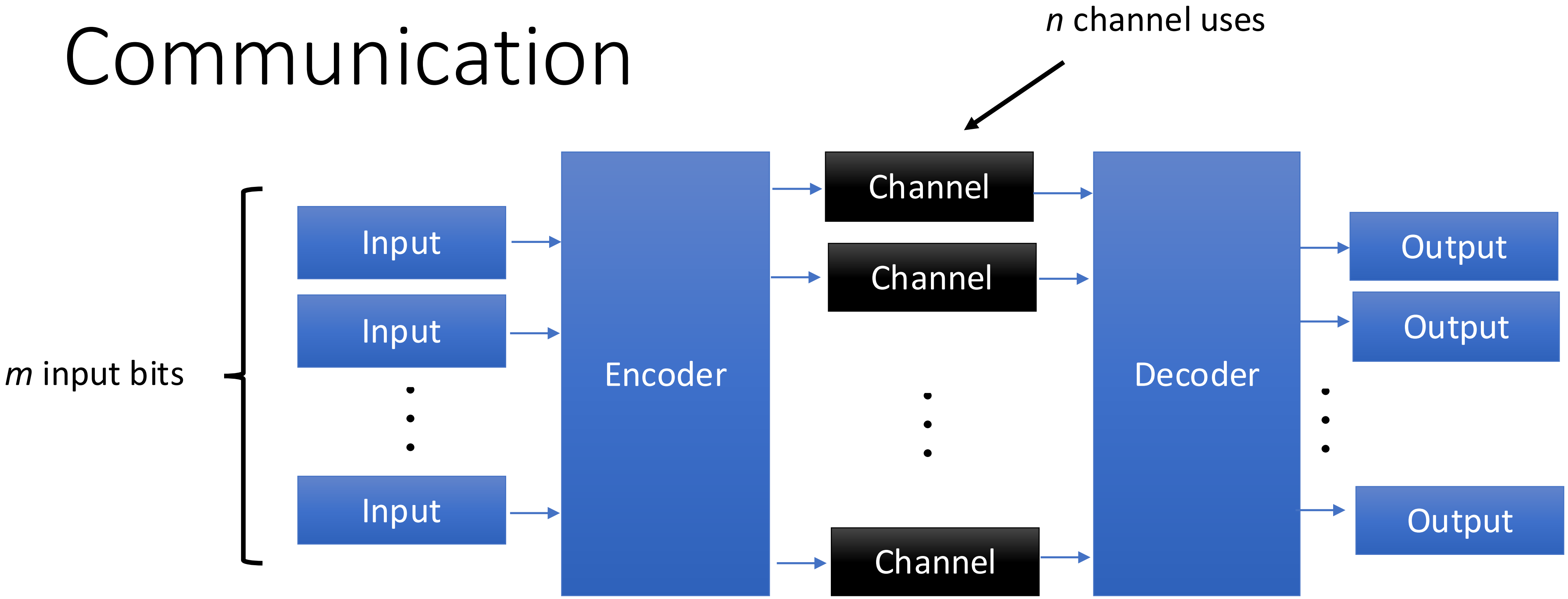
Application:
Fault tolerant Quantum Communication

Communication

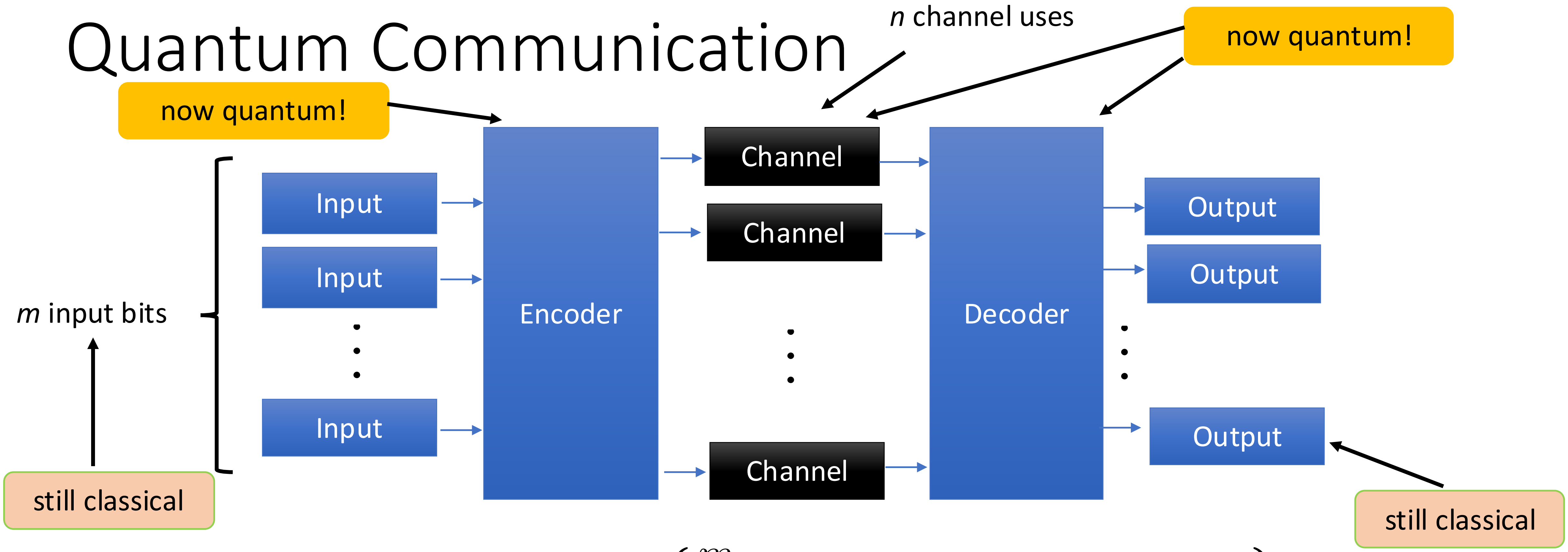


- **Capacity** = $$\lim_{\epsilon\to 0}\lim_{n\to\infty}\sup\left\{\frac{m}{n} : Prob(input \neq output) \leq \epsilon\right\}$$

  = max I(X:Y) bits/channel

H(X)+H(Y)-H(XY) ← Shannon entropy

Quantum Communication



- **Capacity** = $\lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \left\{ \frac{m}{n} : Prob(input \neq output) \leq \epsilon \right\}$

Classical capacity of a quantum channel

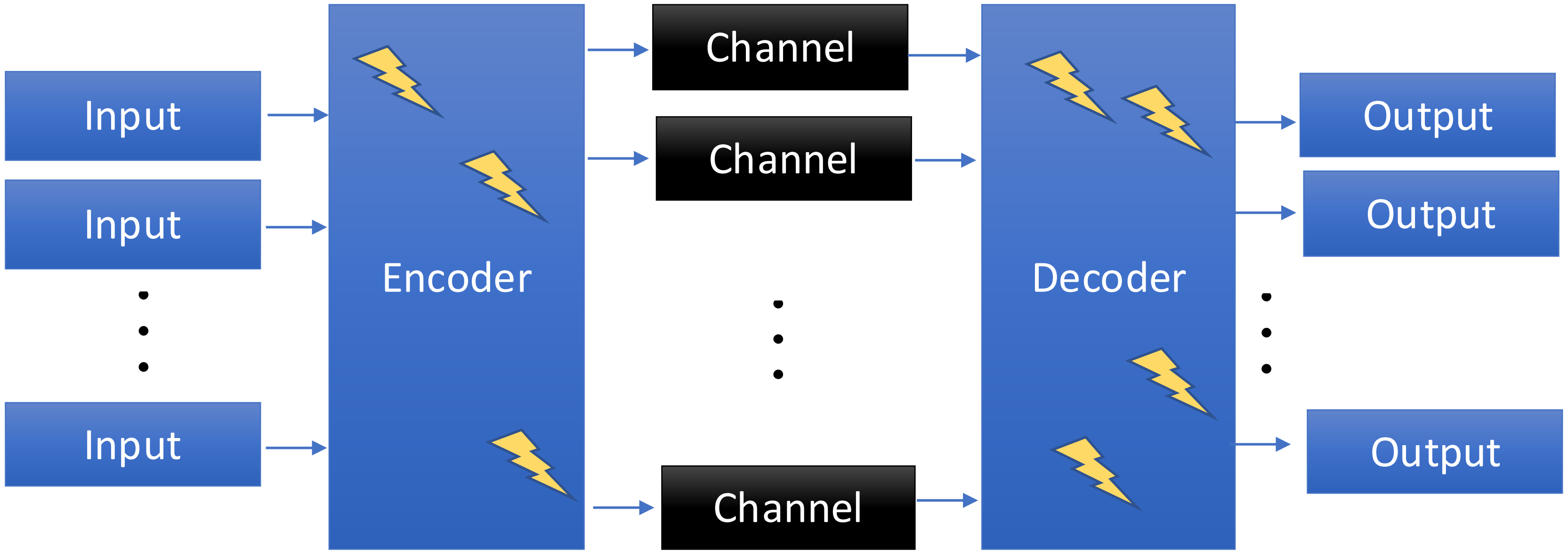
= max I(X:Y) bits/channel + regularization

HSW-theorem

H(X)+H(Y)-H(XY) ← von Neumann entropy

Observation

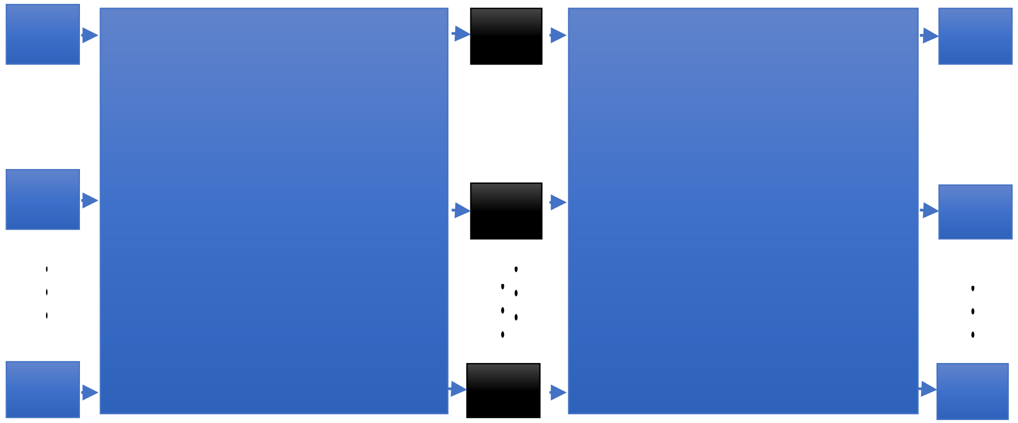
- Assumption of noiseless encoder/decoder unrealistic in quantum case



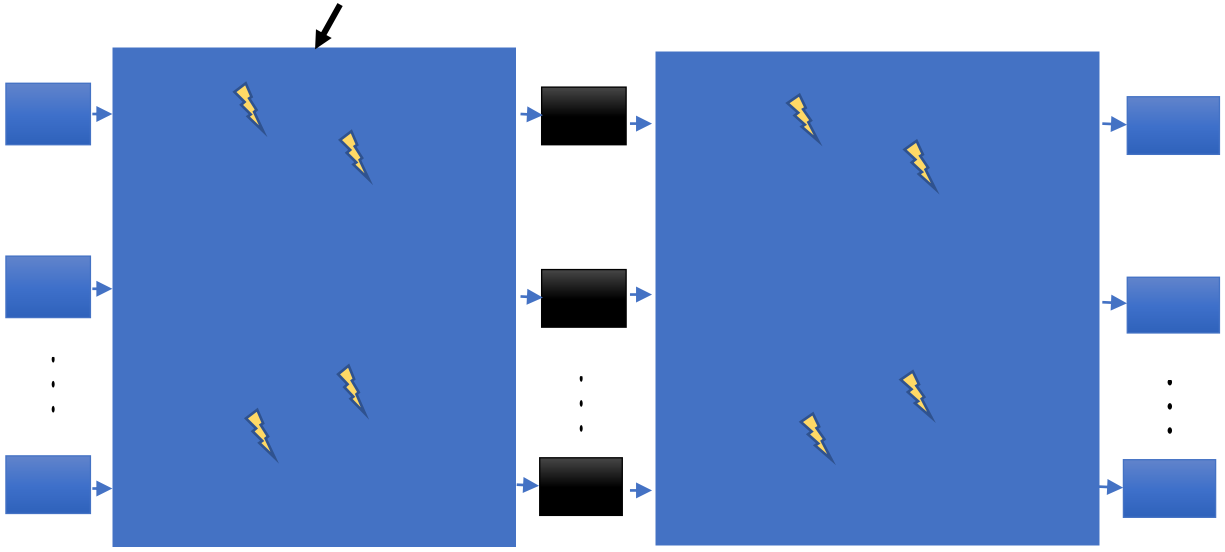
- Entirely new problems appear
  - Is quantum communication possible at all?
  - Is capacity formula continuous? $\delta \to 0$

gate error rate

Fault-tolerant capacity (Christandl, Müller-Hermes, 2024)



noise is not allowed to decrease with growing *n*



Capacity(0-noise)

$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \left\{ \frac{m}{n} : Prob(input \neq output) \leq \epsilon \right\}$$

over encoders-decoder pairs with m input bits

number of channel uses

Capacity(δ -noise)

$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \left\{ \frac{m}{n} : Prob(input \neq output) \leq \epsilon \right\}$$

over (fault-tolerant) encoder-decoder pairs
e.g. over fault-tolerant codes
& encoders-decoder pairs with m input bits

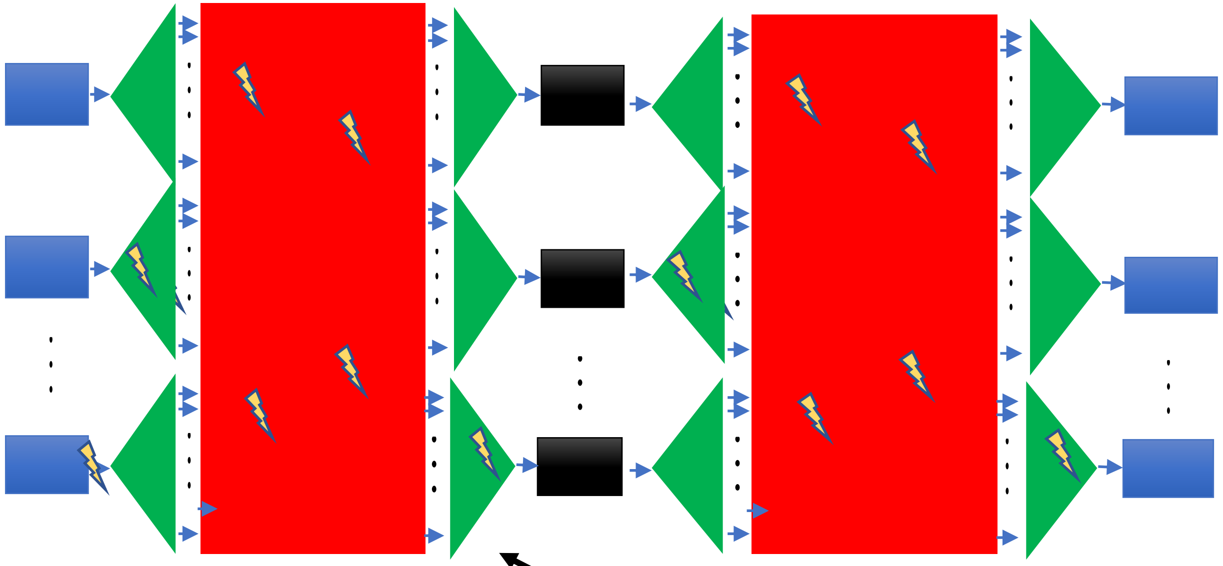
Threshold theorem for Quantum Communication

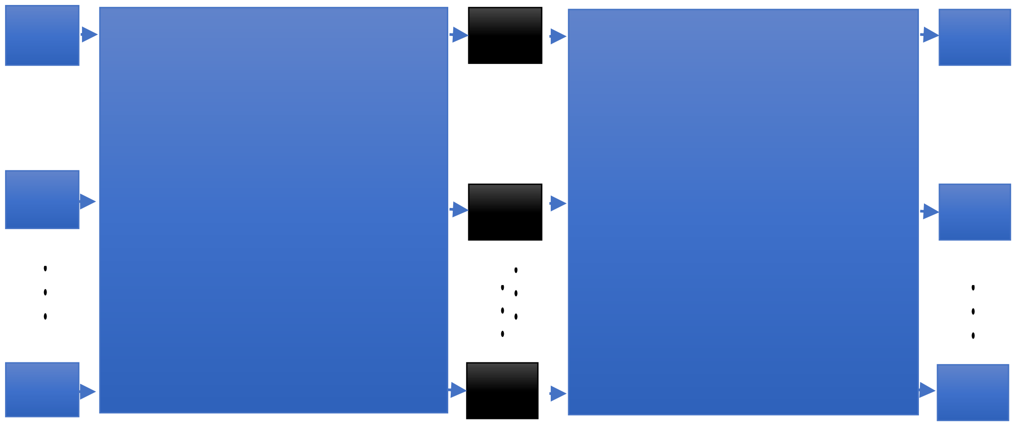
(Christandl, Müller-Hermes, 2024)

$f(\delta) \to 0, \text{ as } \delta \to 0$
can depend on channel

- There is a threshold $\delta_0 > 0$ s.th. for all $\delta \leq \delta_0$

Capacity(δ-noise)>Capacity (0-noise) - $f(\delta)$

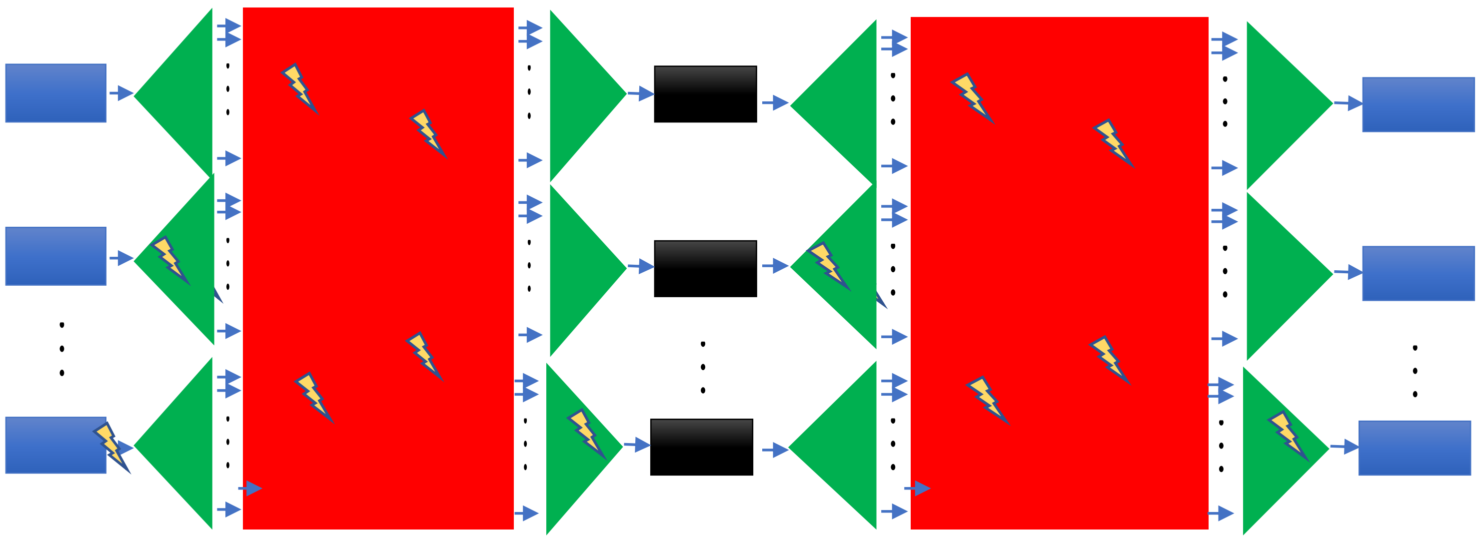


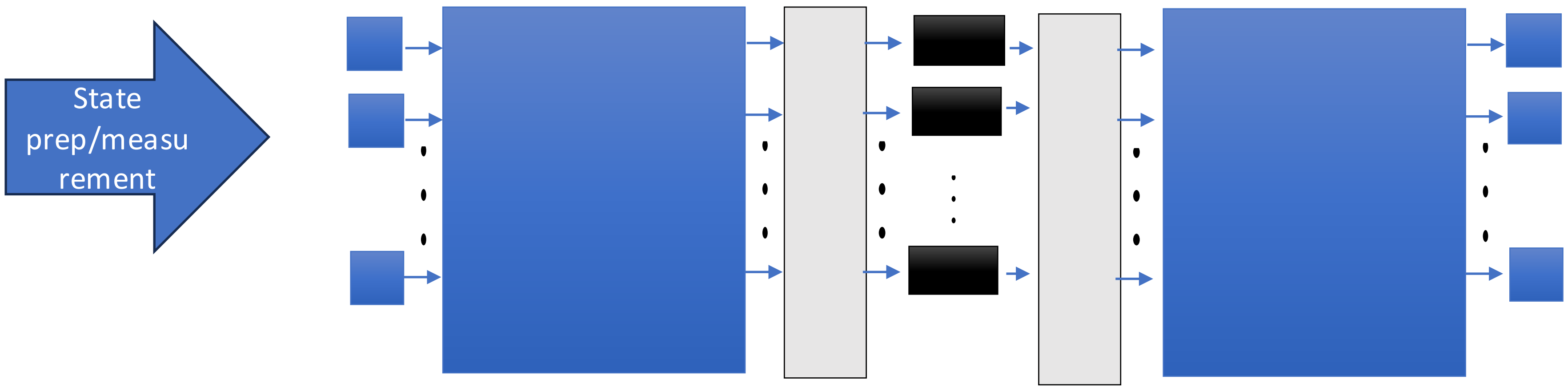


level can depend on *n*

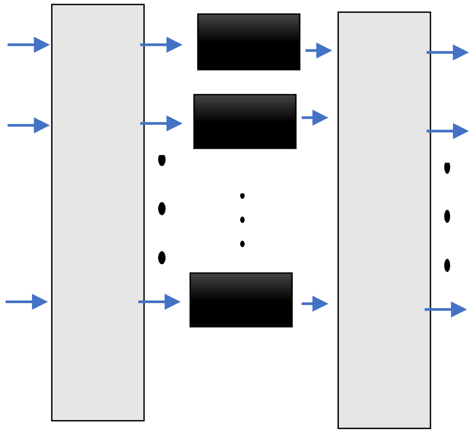
Also for entanglement-assisted capacity
(Belzig, Ch., Müller-Hermes, 2024)
Also code-by-code and for general noise (Ch., Fawzi, Goswami)

Proof: step 1

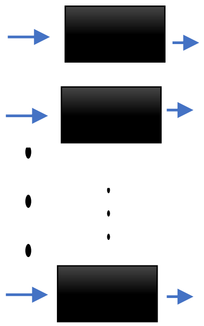




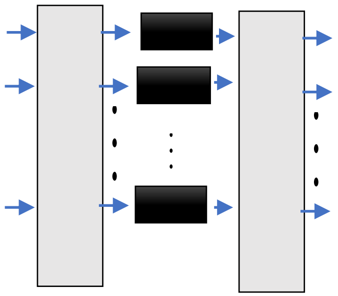
Proof: step 2



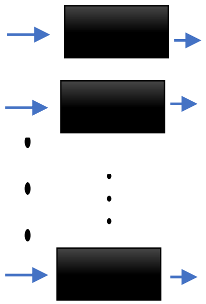
has similar capacity to



Build encoder/decoder for



not for



or consider robust codes

Threshold theorem for Quantum Communication

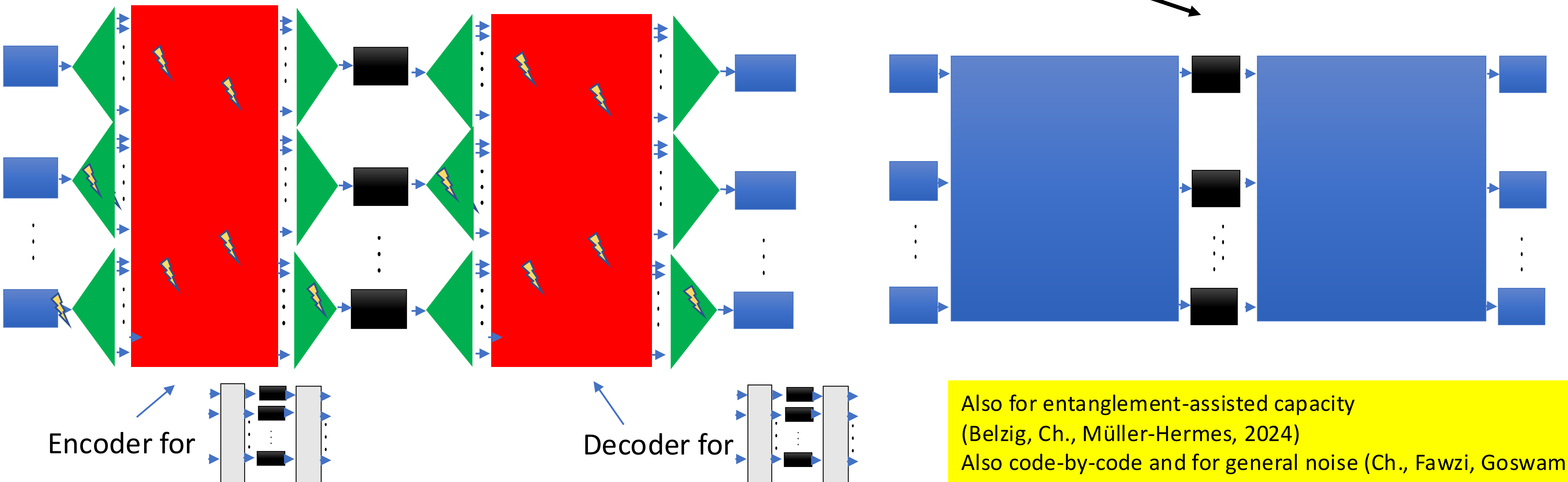
(Christandl, Müller-Hermes, 2024)

$f(\delta) \to 0, \text{ as } \delta \to 0$

can depend on channel

- There is a threshold $\delta_0 > 0$ s.th. for all $\delta \leq \delta_0$

Capacity(δ-noise)>Capacity (0-noise) - $f(\delta)$

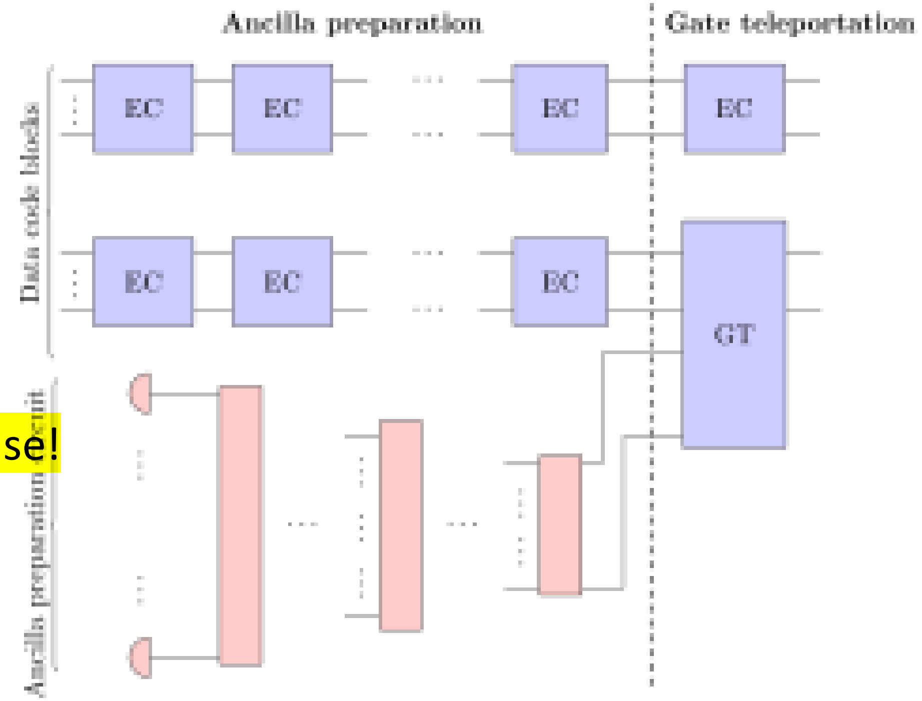


Also for entanglement-assisted capacity
(Belzig, Ch., Müller-Hermes, 2024)
Also code-by-code and for general noise (Ch., Fawzi, Goswami)

Application:
Fault-tolerant Quantum Computation with Constant Overhead for General Noise

Follow Gottesman's blueprint

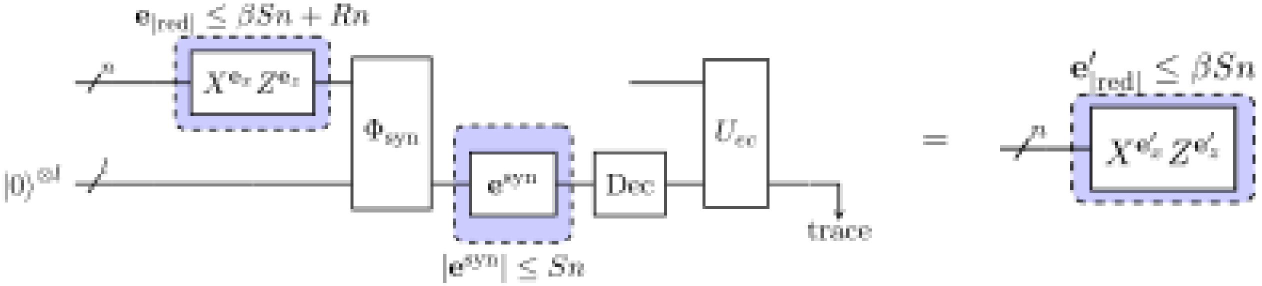
- Serialize circuit (each time step only one gate)
- Encode n data qubits into $\sqrt{n}$ blocks of $\sqrt{n}$ qubits
- Each block in a constant rate quantum LDPC code
- Each gate touches at most two blocks
- Implement via gate teleportation, Now for general noise!
  this requires state preparation!
  some overhead, but only for $2\sqrt{n}$ qubits



- In the meantime, error correction on other blocks
  this requires single shot error correction!

Now for general noise!

Upgrade from Gu et al.
CMP'24



Summary

- New paradigm of fault-tolerant quantum input/output
- Relevance for quantum communication
  - Theoretical: establishes that standard quantum Shannon theory is appropriate model
  - Experimental: gives explicit constructions
- Broader relevance in quantum computing
  - Several small quantum computers connected by noisy communication lines
  - Constant-overhead fault-tolerance for general noise